Mean flows and disturbances

Lagrangian for a one-dimensional fluid:

$$\mathrm{L}\left[x(a,\tau)\right] \equiv \iint d\tau\, da \left\{ \tfrac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 - E\left(\frac{\partial x}{\partial a}\right) \right\}$$

Hamilton's principle:

$$\int \mathrm{d}\tau \quad \mathrm{L}\left[x(a,\tau)\right] = 0 \qquad \text{for arbitrary} \qquad \delta x(a,\tau)$$

Regard $x(a,\tau)$ as a time-dependent mapping:

Now define the composite mapping

$$x(a,\tau) = X(a,\tau) + \xi(X,T)$$

$X(a,\tau)$ is the mean flow.

$\xi(X,T)$ is the displacement of the fluid particle labeled by a from the location it would have if it had moved with the mean flow.

Rewrite the Lagrangian in terms of $X(a,\tau)$ and $\xi(X,T)$.

$$\mathrm{L}\big[x(a,\tau)\big] \equiv \iint d\tau\, da \left\{ \tfrac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 - E\left(\frac{\partial x}{\partial a}\right) \right\}$$

$$x(a,\tau) = X(a,\tau) + \xi(X,T)$$

Rewrite the time derivative as

$$\frac{\partial x}{\partial \tau} = \frac{\partial X}{\partial \tau} + \left(\frac{\partial}{\partial T} + \frac{\partial X}{\partial \tau}\frac{\partial}{\partial X}\right)\xi(X,T)$$

that is

$$u = U + D\xi$$

where

$$U \equiv \frac{\partial X}{\partial \tau} \quad \text{and} \quad D \equiv \frac{\partial}{\partial T} + U\frac{\partial}{\partial X}$$

Rewrite the Jacobian as

$$\frac{\partial x}{\partial a} = \frac{\partial}{\partial a}\big(X(a,\tau) + \xi(X,T)\big) = \frac{\partial X}{\partial a} + \frac{\partial \xi}{\partial X}\frac{\partial X}{\partial a} = V + V\frac{\partial \xi}{\partial X}$$

where

$$V \equiv \frac{\partial X}{\partial a}$$

The Lagrangian becomes

$$\mathrm{L}\big[X(a,\tau),\, \xi(X,T)\big] = \iint d\tau\, da \left\{ \tfrac{1}{2}(U + D\xi)^2 - E\left(V + V\frac{\partial \xi}{\partial X}\right) \right\}$$

$$\mathrm{L}\left[X(a,\tau),\ \xi(X,T)\right] = \iint d\tau\, da \left\{ \tfrac{1}{2}\left(U + D\xi\right)^2 - E\left(V + V\frac{\partial \xi}{\partial X}\right)\right\}$$

Hamilton's principle:

$$\int \mathrm{d}\tau \quad \mathrm{L}\left[X(a,\tau),\xi(X,T)\right] = 0 \qquad \text{for arbitrary} \quad \delta X(a,\tau),\ \delta\xi(X,T)$$

Yields *two* dynamical equations, reflecting the many possible ways of dividing a single flow into a mean flow and a disturbance.

Suppose the disturbance takes the form of a *slowly-varying* wave:

$$\xi(X,T) = A(X,T)\ \cos\theta(X,T)$$

$$k \equiv \frac{\partial \theta}{\partial X}$$

$$\omega \equiv -\frac{\partial \theta}{\partial T}$$

Assume also that A is small. Then

$$E\left(V + V\frac{\partial \xi}{\partial X}\right) = E(V) + E'(V)V\frac{\partial \xi}{\partial X} + \tfrac{1}{2}E''(V)\left(V\frac{\partial \xi}{\partial X}\right)^2 + O\left(\xi^3\right)$$

$$\mathrm{L} = \mathrm{L}_1\left[X(a,\tau)\right] + \mathrm{L}_2\left[X(a,\tau),\ A(X,T),\ \theta(X,T)\right]$$

$$\mathrm{L}_1 = \iint d\tau\, da\ \left\{\tfrac{1}{2}U^2 - E(V)\right\}$$

$$\mathrm{L}_2 = \iint d\tau\, da\ \left\{U D\xi + \tfrac{1}{2}\left(D\xi\right)^2 - E'(V)V\frac{\partial \xi}{\partial X} - \tfrac{1}{2}E''(V)\left(V\frac{\partial \xi}{\partial X}\right)^2\right\}$$

Use the slowly varying approximation to simplify

$$\mathrm{L}_2 = \iint d\tau\, da \left\{ U\,D\xi + \tfrac{1}{2}(D\xi)^2 - E'(V)V\frac{\partial \xi}{\partial X} - \tfrac{1}{2}E''(V)\left(V\frac{\partial \xi}{\partial X}\right)^2 \right\}$$

e.g.

$$\begin{aligned}\mathrm{L}_2 &= \iint d\tau\, da \left\{ -\tfrac{1}{2}E''(V)\left(V\frac{\partial \xi}{\partial X}\right)^2 \right\} \\ &\approx \iint d\tau\, da \left\{ -\tfrac{1}{2}E''(V)(V\,A\,k\sin\theta)^2 \right\} \\ &\approx \iint d\tau\, da \left\{ -\tfrac{1}{4}E''(V)(V\,A\,k)^2 \right\}\end{aligned}$$

The result is the *averaged Lagrangian*:

$$\mathrm{L}_2\left[X(a,\tau),\, A(X,T),\, \theta(X,T)\right] = \iint d\tau\, da\ \tfrac{1}{4}A^2\left\{(\omega - Uk)^2 - c^2k^2\right\}$$

where

$$c^2 \equiv V^2E''(V), \qquad k \equiv \frac{\partial \theta}{\partial X}, \qquad \omega \equiv -\frac{\partial \theta}{\partial T}$$

The amplitude variation

$$\delta A: \qquad (\omega - Uk)^2 = c^2k^2$$

yields the dispersion relation

$$\omega = Uk + ck$$

$$\mathrm{L}_2\left[X(a,\tau),\, A(X,T),\, \theta(X,T)\right] = \iint d\tau\, da\; \tfrac{1}{4} A^2 \left\{ (\omega - Uk)^2 - c^2 k^2 \right\}$$

The phase variation

$$\delta \mathrm{L}_2 = \iint dT\, dX \frac{\partial a}{\partial X} \;\{\; \tfrac{1}{2} A^2 \left[(\omega - Uk)\left(- \frac{\partial \delta\theta}{\partial T} - U \frac{\partial \delta\theta}{\partial X} \right) - c^2 k \frac{\partial \delta\theta}{\partial X} \right] \}$$

yields the equation

$$\frac{\partial}{\partial T}(W) + \frac{\partial}{\partial X}\left[(U + c) W \right] = 0$$

for the conservation of *wave action*,

$$W \equiv \frac{E_r}{\omega_r}$$

where

$$E_r = \tfrac{1}{2} \bar{\rho} A^2 \omega_r{}^2$$

is the wave energy in a reference frame moving with the mean flow, and

$$\omega_r = \omega - Uk = ck$$

is the frequency in that same reference frame.

This is Whitham’s averaged Lagrangian approach.

To complete the description of the wave field, we must develop evolution equations for k and ω.

From the definitions

$$k \equiv \frac{\partial \theta}{\partial X} \quad \text{and} \quad \omega \equiv -\frac{\partial \theta}{\partial T}$$

we obtain the consistency equation

$$\frac{\partial k}{\partial T} + \frac{\partial \omega}{\partial X} = 0$$

Substituting from the dispersion relation

$$\omega = (U + c)k$$

we obtain the refraction equation

$$\left[\frac{\partial}{\partial T} + (U + c)\frac{\partial}{\partial X}\right]k = -k\frac{\partial}{\partial X}(U + c)$$

Similarly

$$\frac{\partial \omega}{\partial T} = (U + c)\frac{\partial k}{\partial T} + k\frac{\partial}{\partial T}(U + c)$$

$$= -(U + c)\frac{\partial \omega}{\partial X} + k\frac{\partial}{\partial T}(U + c)$$

i.e.

$$\left[\frac{\partial}{\partial T} + (U + c)\frac{\partial}{\partial X}\right]\omega = +k\frac{\partial}{\partial T}(U + c)$$

This is standard ray theory.
The description of the wave field is complete.

Equations for the mean flow

We get the mean flow equations by varying $X(a,\tau)$.

We have

$$\mathrm{L} = \mathrm{L}_1\left[X(a,\tau)\right] + \mathrm{L}_2\left[X(a,\tau),\, A(X,T),\, \theta(X,T)\right]$$

$$\mathrm{L}_1 = \iint d\tau\, da\, \left\{\tfrac{1}{2}U^2 - E(V)\right\}$$

$$\mathrm{L}_2 = \iint d\tau\, da\, \tfrac{1}{4}A^2\left\{(\omega - Uk)^2 - c^2k^2\right\}$$

where

$$U \equiv \frac{\partial X}{\partial \tau}, \qquad V \equiv \frac{\partial X}{\partial a}$$

Although A and θ are not varied, they are affected by the variations in X. For example:

$$\delta X(a,\tau): \qquad \delta A(X,T) = \frac{\partial A}{\partial X}\delta X(a,\tau)$$

We find

$$\delta \mathrm{L}_1 = \iint d\tau\, da\, \left\{-\frac{\partial^2 X}{\partial \tau^2} - V\frac{\partial P}{\partial X}\right\}\delta X(a,\tau)$$

where $\qquad P = -E'(V)$

and

$$\delta \mathrm{L}_2 = \iint d\tau\, da\, \{\, \tfrac{1}{2}A\,\delta A\left[(\omega - Uk)^2 - c^2k^2\right]$$
$$+ \tfrac{1}{2}A^2\left[(\omega - Uk)(\delta\omega - k\,\delta U - U\,\delta k) - ck(c\,\delta k + k\,\delta c)\right]\,\}$$

but the coefficient of δA vanishes by the dispersion relation.

Therefore

$$\delta \mathrm{L}_2 = \iint d\tau\, da\ \{\ \tfrac{1}{2} A^2 ck\left[\left(\frac{\partial \omega}{\partial X} - (U+c)\frac{\partial k}{\partial X}\right)\delta X - k\frac{\partial \delta X}{\partial \tau} - kc'\frac{\partial \delta X}{\partial a}\right]\ \}$$

$$= \iint d\tau\, da\ \{\ \tfrac{1}{2} A^2 ck\left(\frac{\partial \omega}{\partial X} - (U+c)\frac{\partial k}{\partial X}\right) + \frac{\partial}{\partial \tau}\left(\tfrac{1}{2} A^2 ck^2\right) + \frac{\partial}{\partial a}\left(\tfrac{1}{2} A^2 k^2 cc'\right)\ \}\delta X$$

so the equation for the mean flow is

$$-\frac{\partial U}{\partial \tau} - V\frac{\partial P}{\partial X} + \tfrac{1}{2} A^2 ck\left(\frac{\partial \omega}{\partial X} - (U+c)\frac{\partial k}{\partial X}\right) + \frac{\partial}{\partial \tau}\left(\tfrac{1}{2} A^2 ck^2\right) + V\frac{\partial}{\partial X}\left(\tfrac{1}{2} A^2 k^2 cc'\right) = 0$$

It can also be written

$$\frac{\partial}{\partial \tau}\left(U - \frac{Wk}{\overline{\rho}}\right) = -\frac{1}{\overline{\rho}}\frac{\partial}{\partial X}\left(P - \tfrac{1}{2} A^2 k^2 cc'\right) + \frac{Wk}{\overline{\rho}}\frac{\partial}{\partial X}(U+c)$$

This equation is equivalent to

$$\frac{\partial}{\partial T}(\overline{\rho} U) + \frac{\partial}{\partial X}\left(\overline{\rho} U^2\right) + \frac{\partial P}{\partial X} = \frac{\partial R}{\partial X}$$

where

$$R = \tfrac{1}{2} A^2 k^2 cc' - \tfrac{1}{2}\overline{\rho} A^2 k^2 c^2$$

is the radiation stress.
With the continuity equation for the mean flow

$$\frac{\partial \overline{\rho}}{\partial \tau} + \overline{\rho}\frac{\partial U}{\partial X} = 0$$

we have a complete description of the mean flow and the wave field.

Generalization to 3 dimensions

The method is the same as in the one-dimensional case, but the final result is more intersting

$$\mathrm{L}\left[\mathbf{x}\left(\mathbf{a},\tau\right)\right] = \int d\tau \iiint d\mathbf{a}\left\{\tfrac{1}{2}\frac{\partial \mathbf{x}}{\partial \tau}\cdot\frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\right)\right\}$$

$$\mathbf{x}(\mathbf{a},\tau) = \mathbf{X}(\mathbf{a},\tau) + \boldsymbol{\xi}(\mathbf{X},T),$$

$$\frac{\partial \mathbf{x}}{\partial \tau} = \frac{\partial \mathbf{X}}{\partial \tau} + \left(\frac{\partial}{\partial T} + \frac{\partial \mathbf{X}}{\partial \tau}\cdot\nabla_{\mathbf{X}}\right)\boldsymbol{\xi} \equiv \mathbf{U} + D\boldsymbol{\xi}\ ,$$

$$E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}\right) = E(V) + VE'(V)\ \{\nabla_{\mathbf{X}}\cdot\boldsymbol{\xi} + \frac{\partial(\eta,\zeta)}{\partial(Y,Z)} + \frac{\partial(\xi,\zeta)}{\partial(X,Z)} + \frac{\partial(\xi,\eta)}{\partial(X,Y)}\}$$
$$+\ \tfrac{1}{2}c^2(\nabla_{\mathbf{X}}\cdot\boldsymbol{\xi})^2 + O(\boldsymbol{\xi}^3)$$

$$\boldsymbol{\xi} = Re\ (\mathbf{A}\ e^{i\theta})$$

$$\mathrm{L} = \mathrm{L}_1[\mathbf{X}(\mathbf{a},\tau)] + \mathrm{L}_2[\mathbf{X}(\mathbf{a},\tau),\ \mathbf{A}(\mathbf{X},T),\ \theta(\mathbf{X},T)],$$

$$\mathrm{L}_1 = \int d\tau \iiint d\mathbf{a}\left\{\tfrac{1}{2}\frac{\partial \mathbf{X}}{\partial \tau}\cdot\frac{\partial \mathbf{X}}{\partial \tau} - E\left(\frac{\partial(\mathbf{X})}{\partial(\mathbf{a})}\right)\right\}$$

$$\mathrm{L}_2 = \int d\tau \iiint d\mathbf{a}\left\{\tfrac{1}{4}\left(\omega - \mathbf{U}\cdot\mathbf{k}\right)^2\mathbf{A}\cdot\mathbf{A}^* - \tfrac{1}{4}c^2\left(\mathbf{k}\cdot\mathbf{A}\right)\left(\mathbf{k}\cdot\mathbf{A}^*\right)\right\}$$

$$\mathbf{k} \equiv \nabla_{\mathbf{X}}\theta \qquad \text{and} \qquad \omega \equiv -\frac{\partial \theta}{\partial T}$$

Derivation of the wave equations

$$\mathrm{L}_2 = \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{4}\left(\omega - \mathbf{U}\cdot\mathbf{k}\right)^2 \mathbf{A}\cdot\mathbf{A}^* - \tfrac{1}{4}c^2\left(\mathbf{k}\cdot\mathbf{A}\right)\left(\mathbf{k}\cdot\mathbf{A}^*\right)\right\}$$

Amplitude variation

$$\delta\mathbf{A}: \qquad \mathbf{A} = A\frac{\mathbf{k}}{|\mathbf{k}|} \qquad \text{and} \qquad \omega_r \equiv \omega - \mathbf{U}\cdot\mathbf{k} = ck$$

which allows us to simplify:

$$\mathrm{L}_2 = \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{4}\left(\omega - \mathbf{U}\cdot\mathbf{k}\right)^2 A^2 - \tfrac{1}{4}c^2 k^2 A^2 \right\}$$

Phase variation

$$\delta\theta: \qquad \frac{\partial W}{\partial T} + \nabla_{\mathbf{X}}\cdot\left[\left(\mathbf{U} + c\frac{\mathbf{k}}{k}\right)W\right] = 0$$

yields equation for the wave action,

$$W = \tfrac{1}{2}\bar{\rho}A^2 ck = \frac{E_r}{\omega_r}$$

The dispersion relation and the refraction equation,

$$\frac{\partial\mathbf{k}}{\partial T} + \nabla_{\mathbf{X}}\left(\mathbf{U}\cdot\mathbf{k} + ck\right) = 0,$$

complete the description of the wave field.

Derivation of the equations for the mean flow

$$\delta \mathrm{L}_2 = \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{2} A^2 \omega_r \left(\delta\omega - \mathbf{U}\cdot\delta\mathbf{k} - \mathbf{k}\cdot\frac{\partial \delta\mathbf{X}}{\partial \tau} \right) - \tfrac{1}{2} A^2 ck \left(c\,\delta k + kc'\,\delta\frac{\partial(\mathbf{X})}{\partial(\mathbf{a})} \right) \right\}$$

Integrating by parts, and combining this with the result from varying L1, we obtain

$$\delta\mathbf{X}: \qquad \frac{\partial}{\partial \tau}\left(U_i - \frac{Wk_i}{\overline{\rho}} \right) = -\frac{1}{\overline{\rho}}\frac{\partial}{\partial X_i}\left(P - \tfrac{1}{2} A^2 k^2 cc' \right) + \frac{W}{\overline{\rho}}\left(k_j \frac{\partial U_j}{\partial X_i} + k\frac{\partial c}{\partial X_i} \right)$$

This can be manipulated to give the complete set of mean flow equations

$$\frac{\partial}{\partial T}\left(\overline{\rho} U_i\right) + \frac{\partial}{\partial X_i}\left(\overline{\rho} U_i U_j\right) + \frac{\partial P}{\partial X_i} = \frac{\partial R_{ij}}{\partial X_j}$$

$$R_{ij} = \tfrac{1}{2} A^2 \left(cc' k^2 \delta_{ij} - \overline{\rho} c^2 k_i k_j \right)$$

$$\frac{\partial}{\partial T}\left(\overline{\rho}\right) + \frac{\partial}{\partial X_i}\left(\overline{\rho} U_i\right) = 0$$

and wave equations

$$\frac{\partial W}{\partial T} + \nabla_{\mathbf{X}} \cdot \left[\left(\mathbf{U} + c\frac{\mathbf{k}}{k} \right) W \right] = 0 \qquad W \equiv \tfrac{1}{2}\overline{\rho} A^2 ck = \frac{E_r}{\omega_r}$$

$$\omega_r \equiv \omega - \mathbf{U}\cdot\mathbf{k} = ck$$

$$\frac{\partial \mathbf{k}}{\partial T} + \nabla_{\mathbf{X}}\left(\mathbf{U}\cdot\mathbf{k} + ck \right) = 0$$

However, the "raw equation"

$$\delta\mathbf{X}: \qquad \frac{\partial}{\partial\tau}\left(U_i - \frac{Wk_i}{\bar{\rho}}\right) = -\frac{1}{\bar{\rho}}\frac{\partial}{\partial X_i}\left(P - \tfrac{1}{2}A^2k^2cc'\right) + \frac{W}{\bar{\rho}}\left(k_j\frac{\partial U_j}{\partial X_i} + k\frac{\partial c}{\partial X_i}\right)$$

leads more directly to the interesting result:

$$\frac{\partial}{\partial\tau}\oint\left(\mathbf{U} - \frac{W\mathbf{k}}{\bar{\rho}}\right)\cdot d\mathbf{X} = 0$$

This reminds us of the (homentropic) vorticity theorem

$$\frac{\partial}{\partial\tau}\oint\mathbf{u}\cdot d\mathbf{x} = 0$$

The latter was associated with the particle-relabeling symmetry. Can we derive the former from this same symmetry?

Particle-relabeling symmetry for the mean flow

The complete Lagrangian is:

$$\mathrm{L} = \int d\tau \iiint d\mathbf{a}\left\{ \tfrac{1}{2}\frac{\partial \mathbf{X}}{\partial \tau}\cdot\frac{\partial \mathbf{X}}{\partial \tau} - E\left(\frac{\partial(\mathbf{X})}{\partial(\mathbf{a})}\right)\right\}$$
$$+ \int d\tau \iiint d\mathbf{a}\left\{ \tfrac{1}{4}\left(\omega - \frac{\partial \mathbf{X}}{\partial \tau}\cdot\mathbf{k}\right)^2 A^2 - \tfrac{1}{4}c^2k^2A^2\right\}$$

Consider particle-label variations that leave $\partial(\mathbf{X})/\partial(\mathbf{a})$ unchanged. These only affect

$$\delta\frac{\partial \mathbf{X}}{\partial \tau}$$

Therefore

$$\delta\mathrm{L} = \int d\tau \iiint d\mathbf{a}\left\{ \frac{\partial \mathbf{X}}{\partial \tau}\cdot\delta\frac{\partial \mathbf{X}}{\partial \tau} - \tfrac{1}{2}A^2\omega_r\mathbf{k}\cdot\delta\frac{\partial \mathbf{X}}{\partial \tau}\right\}$$
$$= \int d\tau \iiint d\mathbf{a}\left\{ \frac{\partial \mathbf{X}}{\partial \tau} - \frac{W\mathbf{k}}{\overline{\rho}}\right\}\cdot\delta\frac{\partial \mathbf{X}}{\partial \tau}$$

Proceeding just as before we obtain

$$\frac{\partial}{\partial \tau}\left(\nabla_{\mathbf{a}}\times\mathbf{A}\right) = 0$$

where now

$$A_j \equiv \left(\frac{\partial X_i}{\partial \tau} - \frac{Wk_i}{\overline{\rho}}\right)\frac{\partial X_i}{\partial a_j}$$

That is

$$\mathbf{A}\cdot d\mathbf{a} = \left\{\frac{\partial \mathbf{X}}{\partial \tau} - \frac{W\mathbf{k}}{\bar{\rho}}\right\}\cdot d\mathbf{X}$$

Converting

$$\frac{\partial}{\partial \tau}\left(\nabla_{\mathbf{a}} \times \mathbf{A}\right) = 0$$

into conventional notation, we have

$$\frac{\partial}{\partial \tau}\left[\frac{\nabla_{\mathbf{X}} \times \left(\mathbf{U} - W\mathbf{k}/\bar{\rho}\right)\cdot \nabla_{\mathbf{X}}\Theta}{\bar{\rho}}\right] = 0$$

This seems to hold for every type of wave, and it seems to be the most general type of conservation law for mean flows in the presence of waves.

Generalizations:

Waves of finite-amplitude

Disturbances of any form. Introduce the ensemble parameter μ

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{X}, T, \mu)$$

and average over μ to obtain the "averaged Lagrangian". This leads to the *generalized Lagrangian mean* formalism of Andrews & McIntyre.