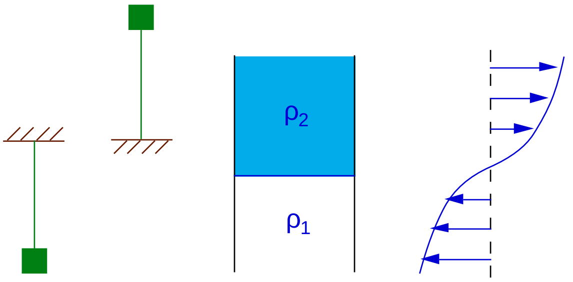
2. STABILITY THEORY

2.1

1. Equilibrium states

When analyzing physical systems, we often start by seeking steady (time-independent) states.

Examples



1. PENDULUM 2. SUPERPOSED FLUIDS 3. SHEAR FLOW

Steady (equilibrium) states are possible solutions of the full time-dependent equations governing the evolution of the system.

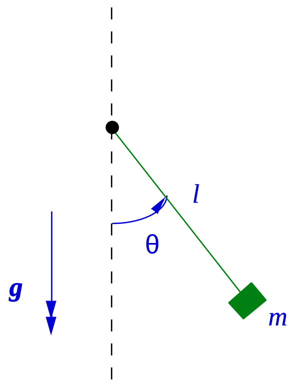
But steady states may be UNSTABLE. That is, any small perturbation (deviation) from the equilibrium state will grow in amplitude.

Such a state is therefore a highly improbable configuration of the physical system.

2. Stability of a Simple Pendulum

2.2

(System with one degree of freedom)



$$ml^2 \frac{d^2\theta}{dt^2} = -mgl\sin\theta - kl\frac{d\theta}{dt}$$

acceleration — restoring moment — viscous damping

Dimensional analysis 2.3

Aim to determine the dimensionless groups of parameters upon which the behaviour of the system depends.

$$[m] = M, \qquad [l] = L\,, \qquad [g] = \frac{L}{T^2}\,, \qquad [k] = \frac{ML}{T}$$

There are two independent time scales

$$t_1 = \sqrt{\frac{l}{g}} \qquad\qquad t_2 = \frac{ml}{k}$$

oscillations   decay

Choose to write $t = \sqrt{\frac{l}{g}}\,\tau$ so that τ is a dimensionless variable.

Then

$$\frac{d^2\theta}{d\tau^2} + \kappa\frac{d\theta}{d\tau} + \sin\theta = 0$$

where $\kappa = \frac{k}{ml}\sqrt{\frac{l}{g}} = \frac{t_1}{t_2}$, the ratio of time scales, is the only parameter governing the evolution of the system.

Equilibrium states 2.4

$$\frac{\partial}{\partial\tau} = 0 \qquad \sin\theta = 0 \qquad \theta = \theta_0 = 0, \pi$$

Equilibrium is independent of κ.

Perturbation

$$\theta = \theta_0 + \varepsilon(\tau) \qquad \ddot{\varepsilon} + \kappa\dot{\varepsilon} + [\cos\theta_0]\sin\varepsilon = 0$$

Linearization

$$\varepsilon << 1 \qquad \sin\varepsilon \approx \varepsilon \qquad \ddot{\varepsilon} + \kappa\dot{\varepsilon} + [\cos\theta_0]\,\varepsilon \approx 0$$

This is a linear equation with constant coefficients

$$\varepsilon \quad e^{\sigma\tau}$$

$$\sigma^2 + \kappa\sigma + \cos\theta_0 = 0$$

$$\boxed{2\sigma = -\kappa \pm \sqrt{\kappa^2 - 4\cos\theta_0}}$$

For each value of θ_0 there are two values of σ (σ_1 and σ_2 say).

2.5

General solution is

$$\varepsilon = Ae^{\sigma_1 t} + Be^{\sigma_2 t}$$

1. $\theta_0 = 0$ $\qquad 2\sigma = -\kappa \pm \sqrt{\kappa^2 - 4}$

   σ_1, σ_2 both negative $\qquad \varepsilon \to 0$ as $t \to \infty$

   System is STABLE

2. $\theta_0 = \pi$ $\qquad 2\sigma = -\kappa \pm \sqrt{\kappa^2 + 4}$

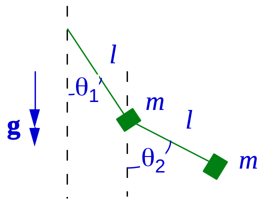
   One of σ_1, σ_2 is positive $\qquad \varepsilon \to \infty$ as $t \to \infty$

   System is UNSTABLE

2.6

3. Stability of a Double Pendulum

(System with two degrees of freedom)

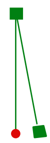


scale time with $\sqrt{\frac{l}{g}}$

$$\ddot{\theta}_1 + \tfrac{1}{2}\cos(\theta_2 - \theta_1)\ddot{\theta}_2 - \tfrac{1}{2}\sin(\theta_2 - \theta_1)\dot{\theta}_2^2 + \sin\theta_1 = 0$$

$$\ddot{\theta}_2 + \cos(\theta_2 - \theta_1)\ddot{\theta}_1 - \sin(\theta_2 - \theta_1)\dot{\theta}_1^2 + \sin\theta_2 = 0$$

Consider equilibrium state $\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\pi\\ 0\end{pmatrix}$



Perturb steady state $\qquad \theta_1 = \pi + \varepsilon_1(t) \qquad \theta_2 = \varepsilon_2(t)$

Linearized perturbation equations $\qquad \ddot{\varepsilon}_1 - \tfrac{1}{2}\ddot{\varepsilon}_2 - \varepsilon_1 = 0 \qquad \ddot{\varepsilon}_2 - \ddot{\varepsilon}_1 + \varepsilon_2 = 0$

4. Normal Modes 2.7

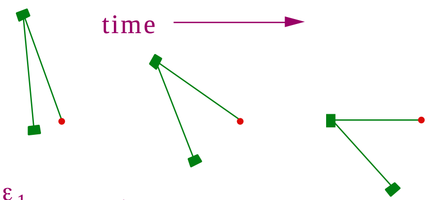
In general, ε_1 and ε_2 are different functions of time. We can find special solutions, called *normal modes*, in which ε_1 and ε_2 have the same time dependence.

$$\varepsilon_1 = a\,f(t)$$
$$\varepsilon_2 = b\,f(t)$$

where a,b are constants.

Normal modes have the property that the shape or configuration of the system doesn't change; only the amplitude changes with time.

<u>Example</u>



The ratio $\dfrac{\varepsilon_1}{\varepsilon_2}$ remains constant.

Since perturbation equations are linear with constant coefficients

$f(t) = e^{\sigma t}$ where σ is the growth rate of the mode.

2.8

Substitute normal-mode solutions into perturbation equations to obtain

$$\sigma^2 a - \tfrac{1}{2}\sigma^2 b - a = 0$$
$$\sigma^2 b - \sigma^2 a + b = 0$$

which can be written in matrix form as

$$\begin{pmatrix} \sigma^2 - 1 & -\tfrac{1}{2}\sigma^2 \\ -\sigma^2 & \sigma^2 + 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \qquad (*)$$

This has non-zero solutions only if the determinant of the matrix is zero

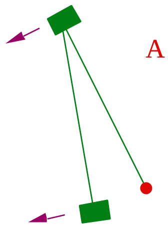
$$\sigma^4 - 1 - \tfrac{1}{2}\sigma^4 = 0$$
$$\underline{\sigma^2 = \pm\sqrt{2}}$$

2.9

1. $\sigma^2 = +\sqrt{2}$ $\qquad f(t) = Ce^{\sqrt[4]{2}\,t} + De^{-\sqrt[4]{2}\,t}$

$f(t) \to \infty$ as $t \to \infty$ so mode is UNSTABLE

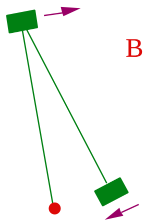
From (*) $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\sqrt{2} \end{pmatrix}$



2. $\sigma^2 = -\sqrt{2}$ $\qquad f(t) = E \sin \sqrt[4]{2}\, t + F \cos \sqrt[4]{2}\, t$

$f(t)$ bounded as $t \to \infty$. Mode is STABLE

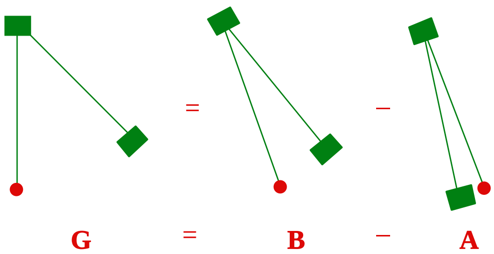
From (*) $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2+\sqrt{2} \end{pmatrix}$



2.10

5. Superposition of Normal Modes

A general solution $\begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix}$ can be written as a sum of normal modes.
For example



Since the amplitude of mode **A** grows in time while the amplitude of mode **B** remains constant, every configuration of the system evolves to look more and more like mode **A**.

When a system is perturbed from an unstable equilibrium, one tends to see the mode with the largest growth rate.

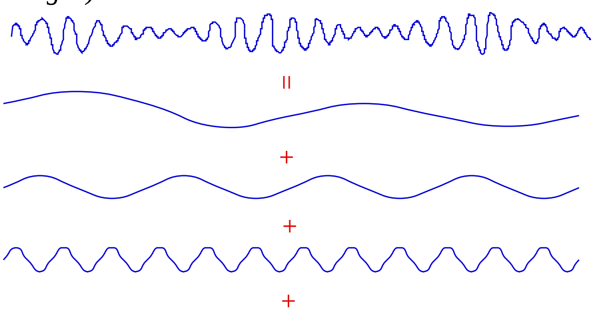
6. Surface-Tension (Rayleigh) Instability of a Cylinder of Fluid

2.11

(System with infinite degrees of freedom.)

Notice from demonstration :
1. A specific spatial structure (wavelength) evolves from random disturbances.
2. Instability of one steady state may lead to another (stable) steady state.

Mode (wavelength) selection



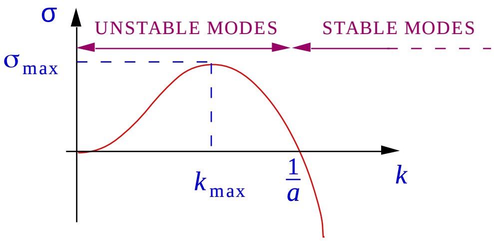
Random disturbances can be expressed as a superposition of pure sinusoidal disturbances (Fourier modes), which are the normal modes of this system.

2.12

There are an infinite number of normal modes - one for each value of the wavenumber

$$k = \frac{2\pi}{\text{wavelength}}$$

Analysis reveals the growth rate $\sigma(k)$ for each normal mode.



All modes with $k < \frac{1}{a}$ (wavelength > $2\pi a$) are unstable, where a is the radius of the undisturbed cylinder.

Will tend to see disturbances with wavenumber $\approx k_{max}$

N.B. Dimensional analysis gives

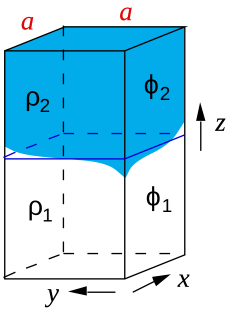
$$\sigma \sim \frac{\gamma}{\mu a}$$

μ : viscosity

γ : surface tension

7. Rayleigh-Taylor Instability of Superposed Fluids

2.13



Dense fluid (ρ_2) lies above lighter fluid (ρ_1) in a square cylinder of side length a. Steady state has interface flat at $z = 0$ and no flow.

Small disturbances described by interface at position $z = \eta(x, y, t)$ and fluid velocity $\mathbf{u} = \nabla\varphi$

(inviscid, irrotational)

Then $\nabla^2\varphi = 0 \qquad \varphi \to 0 \quad \text{as} \quad z \to \mp\infty$

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x} &= 0 \qquad (x = 0, a) \\ \frac{\partial\varphi}{\partial y} &= 0 \qquad (y = 0, a) \\ \frac{\partial\varphi_1}{\partial z} = \frac{\partial\varphi_2}{\partial z} &= \frac{\partial\eta}{\partial t} \qquad (z = \eta) \end{aligned}\right\} \text{kinematic boundary conditions}$$

$$p_1 - p_2 = -\gamma \nabla\cdot\mathbf{n} \quad (z = \eta) \qquad \text{dynamic b.c.}$$

where $\mathbf{n}$ is normal to interface, from fluid 1 to fluid 2, and the pressure is found from Bernouilli. $p + \rho\dfrac{\partial\varphi}{\partial t} + \rho g\eta + \frac{1}{2}\rho(\nabla\varphi)^2 = 0$

8. Scaling and Linearization

2.14

$$(x, y, z) = a\left(x^*, y^*, z^*\right) \quad , \quad t = \sqrt{\frac{\rho_2 a^3}{\gamma}}\, t^* \quad , \quad \varphi = \sqrt{\frac{a\gamma}{\rho_2}}\, \varphi^*$$

The starred variables are dimensionless. Substitute into equations, linearize with $\eta << 1$, $\varphi << 1$ and drop stars.

N.B. In linear problem, interfacial conditions are applied at $z = 0$.

$$\nabla^2\varphi = 0 \qquad \varphi \to 0 \qquad (z \to \pm\infty)$$

$$\frac{\partial\varphi}{\partial x} = 0 \qquad (x = 0, 1)$$

$$\frac{\partial\varphi}{\partial y} = 0 \qquad (y = 0, 1)$$

$$\frac{\partial\varphi_1}{\partial z} = \frac{\partial\varphi_2}{\partial z} \qquad (z = 0)$$

$$\beta\varphi_{1tt} - \varphi_{2tt} = \left(R + \nabla_h^2\right)\varphi_z \qquad (z = 0)$$

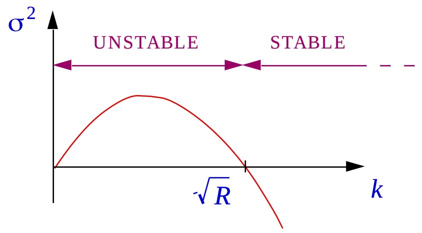
where $\beta = \dfrac{\rho_1}{\rho_2}$, $\nabla_h^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and $R = \dfrac{(\rho_2 - \rho_1)ga^2}{\gamma}$, the Bond number,

is a ratio of the time scales $t_1 = \sqrt{\dfrac{\rho a^3}{\gamma}}$ and $t_2 = \sqrt{\dfrac{a}{g'}}$ $\left[g' = \dfrac{\rho_2 - \rho_1}{\rho_2} g\right]$

Normal modes 2.15

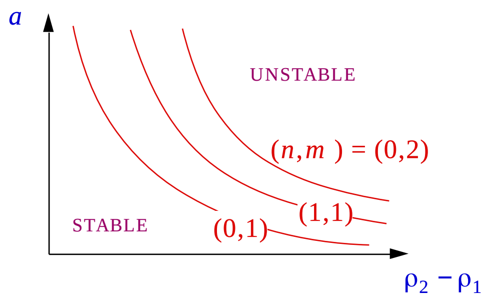
$$\varphi_{1,2} = \pm e^{\sigma t} e^{\pm kz} \cos n\pi x \cos m\pi y \qquad \left[k^2 = \left(n^2 + m^2\right)\pi^2\right]$$

$$\left(1+\beta\right)\sigma^2 = \left(R - k^2\right)k$$



The system is unstable if

$$R > k^2 \qquad a^2 > \frac{\left(n^2 + m^2\right)\pi^2\gamma}{\left(\rho_2 - \rho_1\right) g}$$

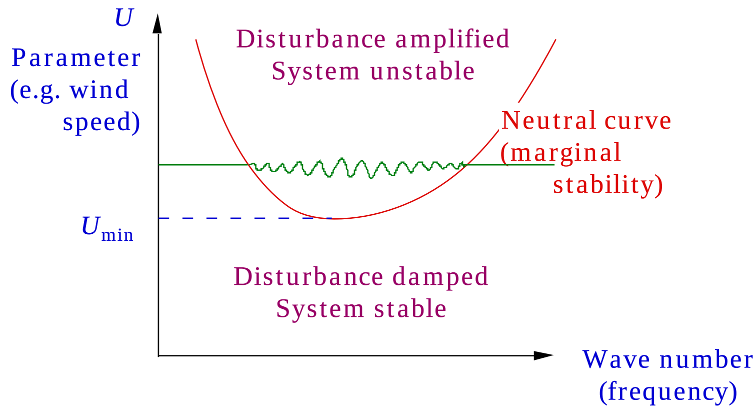


For water over air :
unstable if $a \gtrsim 8.6$ mm

9. Shear-Flow Instability 2.16

FILM DEMONSTRATION - MOLLO CHRISTIANSEN

THE NEUTRAL CURVE



For $U > U_{min}$, there is a band of unstable modes (wavelengths).
Linear theory only tells us that disturbances grow. What they grow into is the subject of nonlinear analyses.

Lecture 2. Stability Theory

Drazin, P.G., 2002 *An Introduction to Hydrodynamic Stability*, Cambridge University Press.

Drazin, P.G. and Howard, L.N., 1966 *Hydrodynamic stability of parallel flow of inviscid fluid*, In: Advances in Applied Mathematics Vol. 9, pp 1-89, Academic Press.

Drazin, P.G. and Reid, W.H., 1991 *Hydrodynamic Stability*, Cambridge University Press.

Chandrasekhar, S., 1961 *Hydrodynamic and Hydromagnetic Stability*, Oxford

Lin, C.C., 1955 *The Theory of Hydrodynamic Stability*, Cambridge University Press.