3. CONVECTION

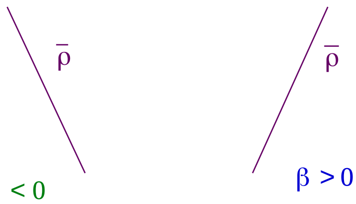
3.1

1. Introduction

Natural convection is the flow of fluid driven by density gradients owing to gradients in temperature, composition...

STATIC STABILITY

There are possible steady states with $\mathbf{u} \equiv 0 \qquad \bar{\rho} = \rho_0 (1 + \beta z)$



$\beta < 0$ — Stable

$\beta > 0$ — Statically unstable
Dynamically unstable?

Important physical parameters

	g	ν	κ	β
Dimensions:	$L T^{-2}$	$L^2 T^{-1}$	$L^2 T^{-1}$	L^{-1}

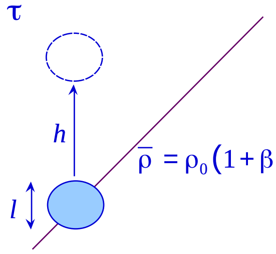
3.2

2. Dimensional analysis

Consider a fluid parcel of characteristic length l elevated through height h in time τ with characteristic velocity $U \sim h/\tau$

Potential energy released

$$E \sim \left(\Delta\rho\, l^3\right) gh$$
$$\sim \rho_0 \beta h\, l^3\; gh$$
$$\sim \rho_0 \beta g h^2 l^3$$



$\bar{\rho} = \rho_0 (1 + \beta z)$

Parcel runs out of buoyancy by diffusion in time $\tau \sim l^2/\kappa$

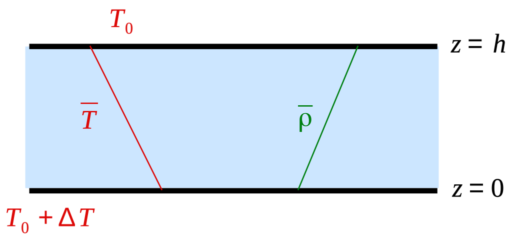
Energy dissipated by viscosity $D \sim \mu \dfrac{U}{l} l^2 . h \sim \mu \dfrac{h^2 l}{\tau} \sim \dfrac{\mu\kappa\, h^2}{l}$

Instability if $E \gtrsim D \qquad \rho_0 \beta g h^2 l^3 \gtrsim \dfrac{\mu\kappa\, h^2}{l}$

i.e. if the Rayleigh number $Ra = \dfrac{\beta g l^4}{\kappa\nu} \gtrsim 1$

3. Rayleigh-Bénard convection

3.3



GOVERNING EQUATIONS

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\,\mathbf{g} + \mu\nabla^2\mathbf{u} \quad \text{(momentum)}$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \qquad \mathbf{g} = (0, 0, -g)$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{(continuity)}$$

$$\rho = \rho_0\left[1 - \alpha\left(T - T_0\right)\right]$$

$$\frac{DT}{Dt} = \kappa\nabla^2 T \quad \text{(diffusion)}$$

STEADY SOLUTIONS

$$\mathbf{u} = 0$$

$$\bar{T} = T_0 + \Delta T\left(1 - \frac{z}{h}\right)$$

$$\bar{\rho} = \rho_0(1 + \beta z)$$

where $\beta = \alpha\frac{\Delta T}{h}$

$$\nabla\bar{p} = \rho_0(1 + \beta z)\,\mathbf{g}$$

4. Linearized perturbation equations

3.4

$$\mathbf{u} = (u, v, w)$$

$$T = \bar{T} + \theta$$

$$p = \bar{p} + p'$$

Substitute into governing equations, linearize in small quantities $\mathbf{u}, \theta, p'$ and non-dimensionalize lengths w.r.t. h, time w.r.t. h^2/κ to obtain

$$\left(\partial_t - \nabla^2\right)\left(P^{-1}\partial_t - \nabla^2\right)\nabla^2 W = Ra\,\nabla_h^2 W$$

where $P = \frac{\nu}{\kappa} = \frac{\text{diffusivity of momentum}}{\text{diffusivity of heat}}$

is the Prandtl number

and $Ra = \frac{\alpha g\,\Delta T\,h^3}{\kappa\nu} = \frac{\beta g h^4}{\kappa\nu}$ is the Rayleigh number.

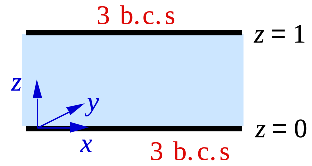
5. Normal modes 3.5

$$w(x, y, z, t) = W(z)\, e^{i(lx+my)+\sigma t}$$

horizontal wave numbers (l, m); growth rate (σ)

Then $\left(D^2 - k^2 - \sigma\right)\left(D^2 - k^2 - P^{-1}\sigma\right)\left(D^2 - k^2\right)W + k^2\, Ra\, W = 0$

BOUNDARY CONDITIONS



i) zero normal velocity: $W = 0$

ii) (a) zero tangential velocity: $DW = 0$

(no slip) $\left(u_x + v_y + w_z\right) = 0$

(b) zero tangential stress: $D^2 W = 0$

$\left(\tau_{xz} = u_z + w_x = 0\right)$ (with w_x struck out and marked 0)

iii) (a) fixed temperature: T = given $\left(D^2 - k^2 - P^{-1}\sigma\right)\left(D^2 - k^2\right)W = 0$

(b) fixed heat flux: $\dfrac{\partial T}{\partial z}$ = given $\left(D^2 - k^2 - P^{-1}\sigma\right)\left(D^2 - k^2\right)DW = 0$

Example 3.6

Stress-free, perfectly conducting boundaries B.C.s (i) (iib) (iiia)

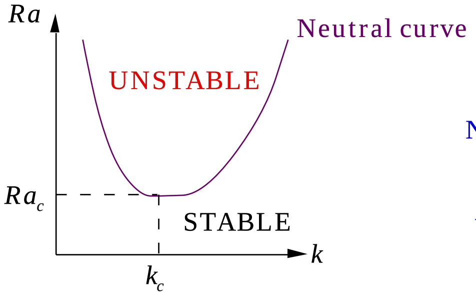
Marginal stability (neutral curve) determined by

$$\left[\left(D^2 - k^2\right)^3 + Ra\, k^2\right]W = 0$$

$$W = D^2 W = \left(D^2 - k^2\right)^2 W = 0 \quad (z = 0, 1)$$

Solution: $W = \sin n\pi z$ with $\left(n^2\pi^2 + k^2\right)^3 = Ra\, k^2$ (Rayleigh 1916)

First unstable mode has n = 1:



Neutral curve: $Ra = \dfrac{\left(k^2 + \pi^2\right)^3}{k^2}$

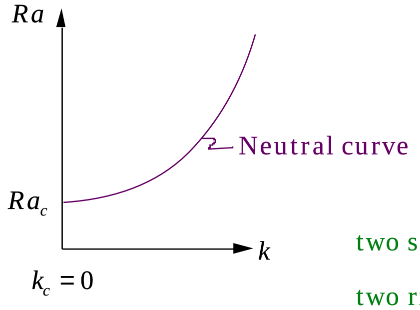
$Ra_c = \dfrac{27\pi^4}{4} \approx 658 \qquad k_c = \dfrac{\pi}{\sqrt{2}} \approx 2.22$

Other results 3.7

one free, one rigid, perfect conductor: $Ra_c \approx 1101$ at $k_c \approx 2.68$

two rigid, perfect conductors: $Ra_c \approx 1708$ at $k_c \approx 3.12$

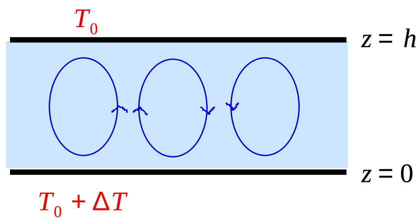
With fixed-heat-flux boundaries:



two stress-free boundaries: $Ra_c = 120$

two rigid boundaries: $Ra_c = 720$

6. Heat flux 3.8



An important quantity in convective flows is the mean (horizontally averaged) heat flux F_H.

This is often measured relative to the conductive heat flux that would occur in the absence of convection in a ratio called the

Nusselt number $Nu = \dfrac{\text{total heat flux}}{\text{conducted heat flux}}$

If the flow is steady then

eddy flux

mean gradient flux

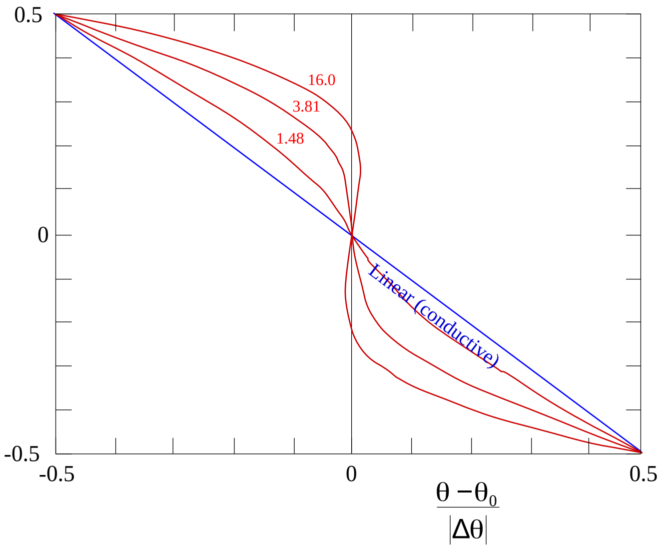
$$Nu = \frac{\overline{WT} - k\overline{T}_z}{k\,\Delta T / h}$$

where $\overline{}$ is a horizontal average

If the flow is unsteady then it is usual to average the Nusselt number temporally as well. N.B. $Nu = 1$ with no flow $Nu > 1$ with flow

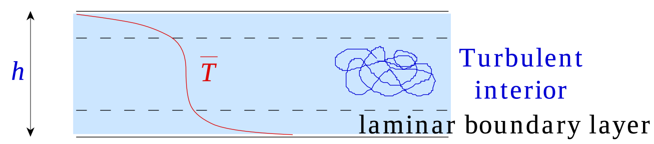
7. Convection at high Rayleigh numbers

3.9



Observed mean temperature profiles in a convecting fluid layer (JST 1979). Note the reversals of gradient at the larger values of $\lambda = Ra/Ra_c$, which are marked on the curves.

3.10



Heat flux $F_H = \frac{k\,\Delta T}{h} Nu$

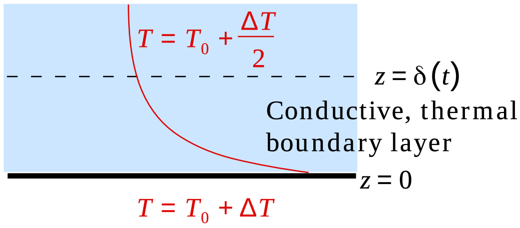
Hypothesis: small-scale plumes detatch from laminar boundary layer only 'see' a statistically uniform, turbulent interior F_H is independent of h

$$Nu = f(Ra, P) \qquad Ra^{1/3} = \left(\frac{\alpha g\,\Delta T\,h^3}{\kappa\nu}\right)^{1/3}$$

$$F_H = c(P)\left(\frac{\alpha g k^2}{\nu}\right)^{1/3} \Delta T^{4/3}$$

This is the so-called famous "four-thirds law" for turbulent heat fluxes.

This picture was formalised by Howard (1966). 3.11



The boundary-layer thickness grows by thermal diffusion with

$$T = T_0 + \frac{\Delta T}{2}\left(1 + \operatorname{erfc}\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\delta = \sqrt{\pi \kappa t}$$

until at time t_c it is unstable and breaks down, reducing δ to zero.

Define t_c by $\frac{\alpha g \Delta T \delta_c^3}{2\kappa\nu} = R_c$

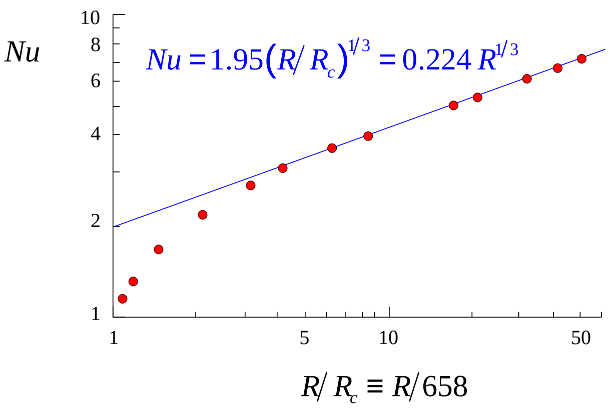
$$\delta_c = \left(\frac{2\kappa\nu}{\alpha g \Delta T}\right)^{1/3} Ra_c^{1/3} \qquad \frac{\delta_c}{h} = \left(\frac{2\,Ra_c}{Ra}\right)^{1/3}$$

The time-averaged heat flux 3.12

$$F_H = \left\langle -k \frac{\partial T}{\partial z}\bigg|_{z=0} \right\rangle = \frac{k\Delta T}{2t_c}\int_0^{t_c} \frac{dt}{\sqrt{\pi\kappa t}} = \frac{k\Delta T}{\delta_c} \qquad Nu = \frac{F_H h}{k \Delta T} = \frac{h}{\delta_c} = \left(\frac{Ra}{2\,Ra_c}\right)^{1/3}$$

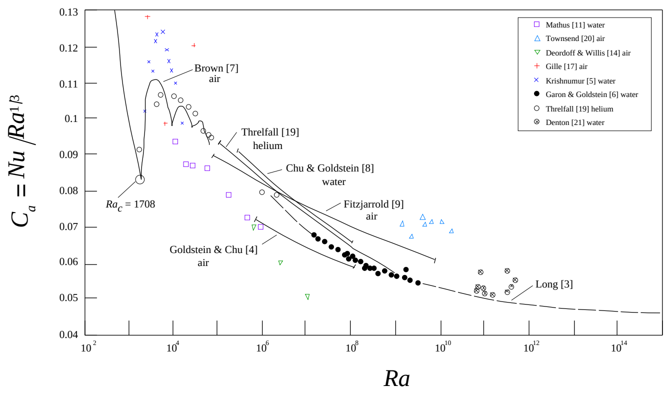
e.g. if $Ra_c = 1100$ then $Nu = 0.077\,Ra^{1/3}$

H^2 (unpublished)



Denton & Wood (1979)

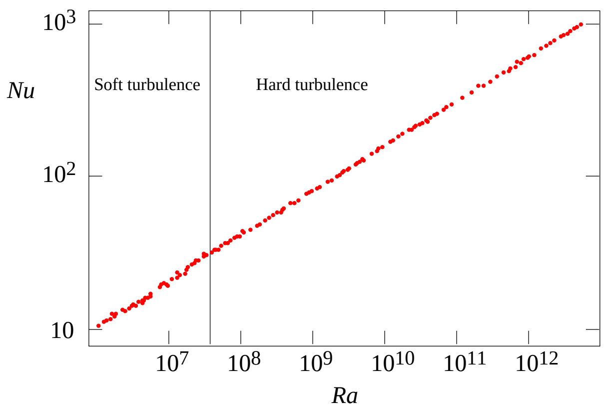
3.13



Recent experiments (Castaing et al.,1989) have found Nu Ra^{β} with

3.14

$$\beta = 0.282 \pm 0.006$$



Double-Diffusive Convection

3.15

8. Double-diffusive convection occurs whenever
 i) two or more components of
 ii) different molecular diffusivities
 iii) contribute in an opposing sense to the vertical density gradient

It is most interesting when
 iv) the mean density field is statically stable

Examples

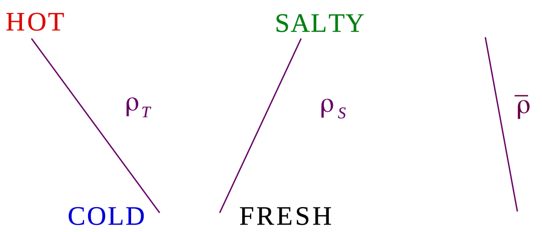
i) Components are heat (T) and salt (S)
salt (T) and sugar (S)
heat (T) and helium (S)

ii) $\tau = \kappa_S / \kappa_T = 10^{-2}$ for salt/heat $= 1/3$ for sugar/salt

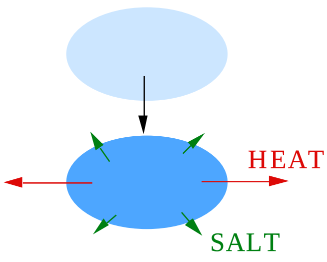
iii)
$$\rho = \rho_0 (1 - \alpha T + \beta S) \qquad \frac{1}{\rho_0}\frac{\partial \rho}{\partial z} = -\alpha \frac{\partial T}{\partial z} + \beta \frac{\partial S}{\partial z}$$

9. Salt Fingers

3.16



Consider a blob of fluid displaced downwards.



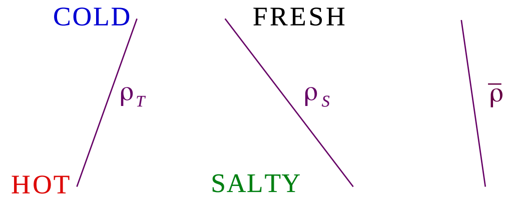
It is hotter and saltier than its new surroundings.

It loses heat quickly by thermal diffusion
It loses salt slowly by molecular diffusion

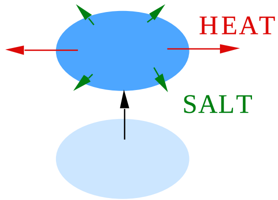
Having lost its thermal buoyancy, it is heavier than its surroundings and sinks.

10. Oscillatory "Diffusive" Convection

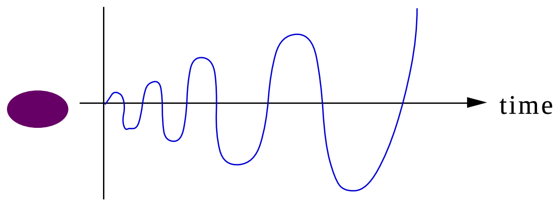
3.17



Consider a blob of fluid displaced upwards.



Having lost its thermal buoyancy it sinks and carries on falling a little past its original position.

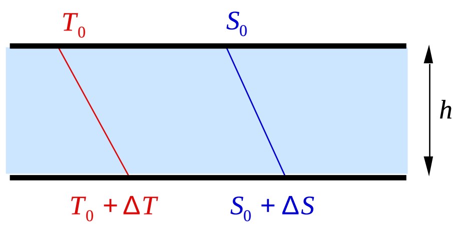


There is an over-stable oscillation.

11. Normal Modes

3.18

It is possible to consider an extended Rayleigh-Bénard problem.



The linearized perturbation equations are

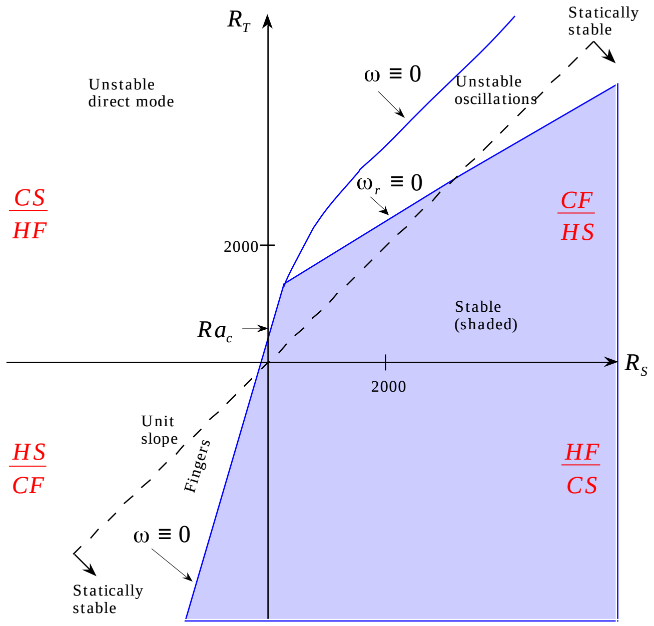
$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial x} = \ {}^2 T \qquad \text{(heat)}$$

$$\frac{\partial S}{\partial t} + \frac{\partial \psi}{\partial x} = \tau \ {}^2 S \qquad \text{(salt)}$$

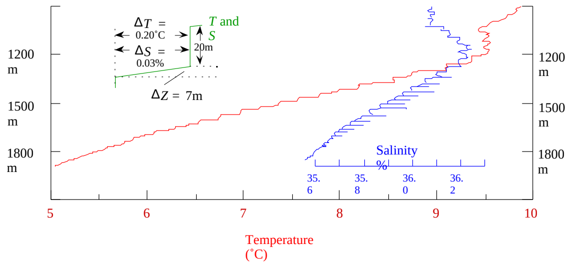
$$P^{-1} \ {}^2 \psi_t = -R_T \frac{\partial T}{\partial x} + R_S \frac{\partial S}{\partial x} + \ {}^4 \psi \qquad \text{(vorticity)}$$

$$R_T = \frac{\alpha g \, \Delta T \, h^3}{\kappa_T \nu}, \quad R_S = \frac{\beta g \, \Delta S \, h^3}{\kappa_T \nu} \quad \Bigg\} \quad \frac{R_S}{R_T} = \frac{\beta \, \Delta S}{\alpha \, \Delta T} = R_\rho \qquad P = \frac{\nu}{\kappa_T} \qquad \tau = \frac{\kappa_S}{\kappa_T}$$

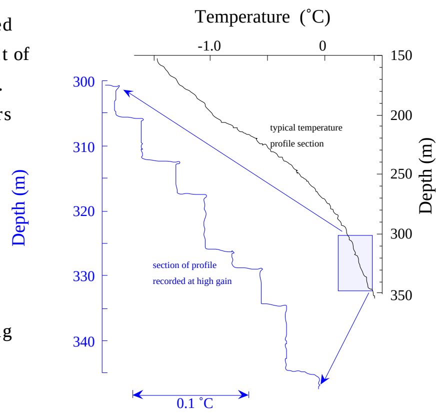
There are normal-mode solutions of the form $f(z)\,e^{\omega t + i\alpha x}$ 3.19



3.20

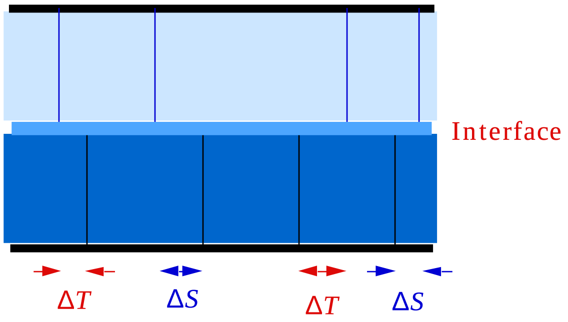


A series of layers in T and S observed under the Mediterranean outflow into the Atlantic, and attributed to the 'finger' mechanism. The sharp spikes to the right of the salinity trace are instrumental and can be ignored. The inset summarizes the mean properties of the layers and interfaces.
(Tait & Howe, 1971)



Temperature profile under Arctic Ice Island T3, showing steps formed by the double-diffusive mechanism.
(Neal, Neshyba & Denner, 1969)

12. Superposed layers 3.21

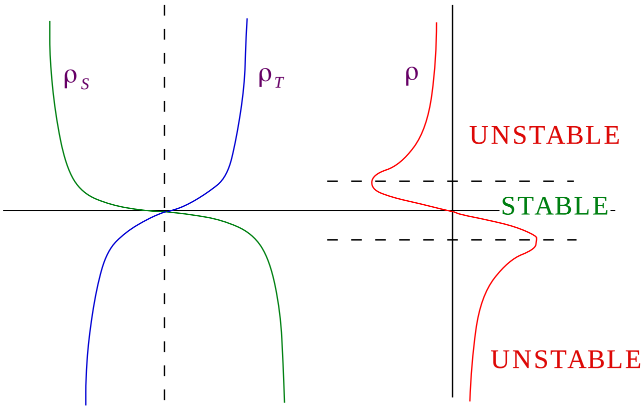


Examples

(i) The Mediterranean outflow into the Atlantic ocean is an intrusion of hot salty water over colder fresher water

(ii) Solar ponds

Diffusive Interface 3.22



The unstable thermal boundary layers either side of the stable intermediate layer break away carrying some salt with them. If the rate of convective salt transport is slower than the rate of diffusion across the intermediate layer then the layer thickens.

At low buoyancy ratios R_ρ, a steady state is possible: the layer is kept thin and the diffusive flux is maintained at a constant high value rather than decaying like $t^{-1/2}$ as it would with no convection.

Parameterisations

3.23

Four-thirds law

$$F_T = \lambda(\tau)\, k \left(\frac{\alpha g}{\kappa \nu}\right)^{1/3} (T_h - T)^{4/3}$$

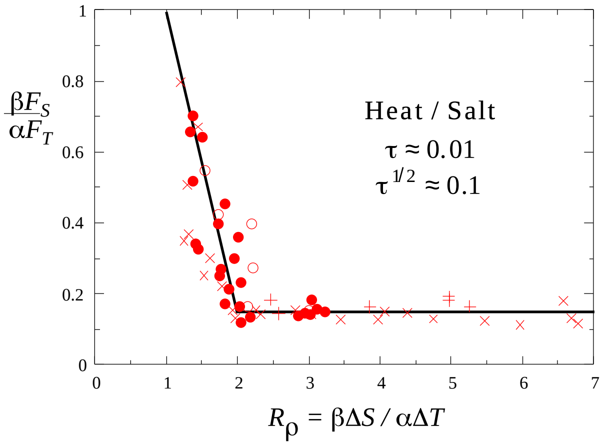
Continuous density

$$\alpha (T_h - T) = \beta (S_h - S)$$

Flux ratio dictated by

diffusive transport

$$\frac{\beta F_S}{\alpha F_T} = \tau^{1/2} \qquad \left[\tau = \frac{D}{\kappa}\right]$$



(Turner, 1965)

Summary of convection

3.24

- $Ra = \frac{\alpha g \, \Delta T \, h^3}{\kappa \nu}$

  $= \frac{\text{destabilizing buoyancy force}}{\text{stabilizing viscous force}}$

- $Ra > Ra_c \sim 10^3$ unstable

- $Ra >> Ra_c$: $Nu = c\, Ra^{1/3}$, $c\, Ra^{2/7}$

  F_H $\Delta T^{4/3}$, $\Delta T^{9/7}$

- Double diffusive convection whenever there are two or more components
- Nonlinear double-diffusive convection inevitably leads to the formation of layers.

Lecture 3. Convection

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