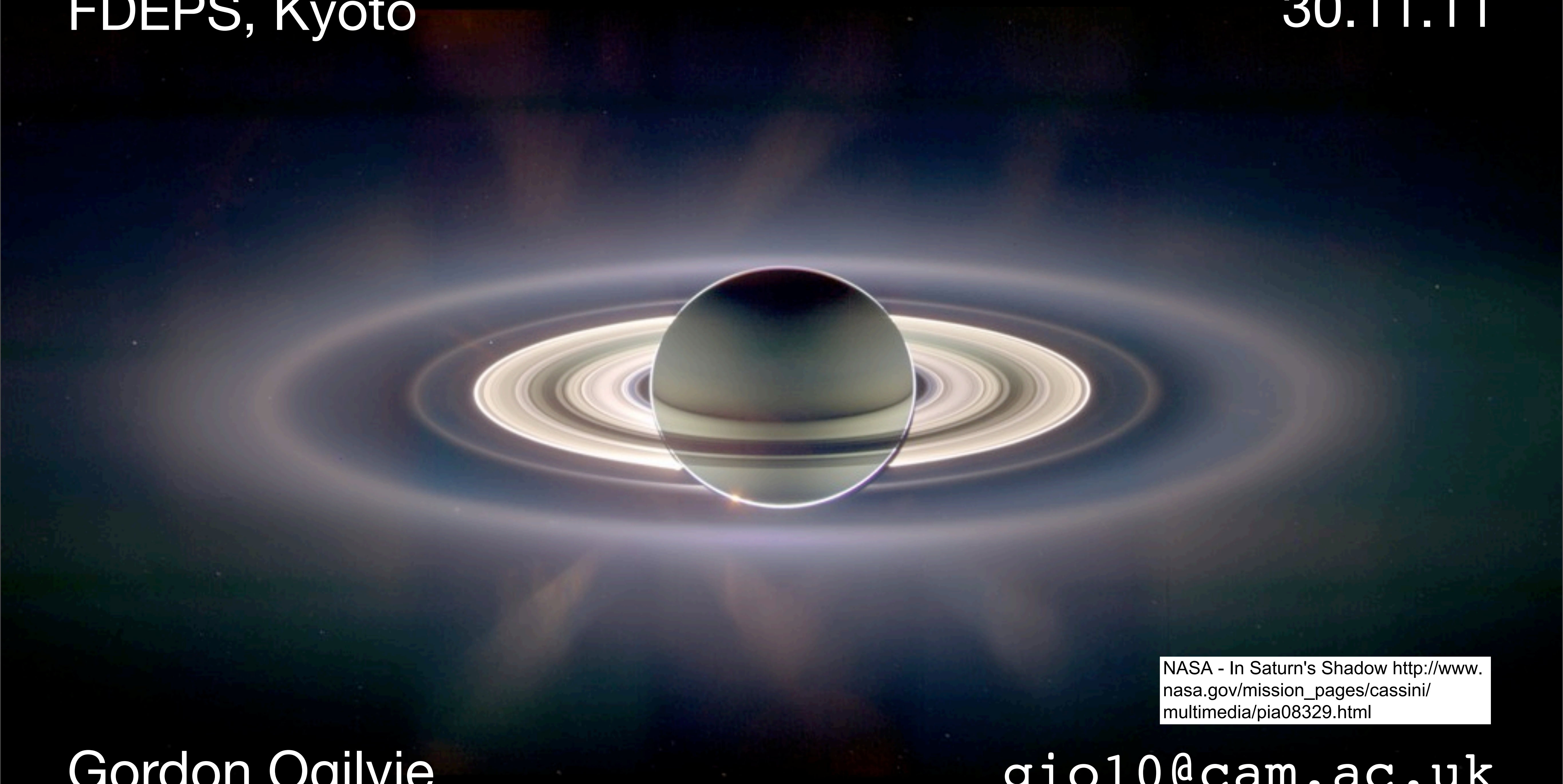
Dynamics of astrophysical discs

Lecture 3

FDEPS, Kyoto

30.11.11



NASA - In Saturn's Shadow http://www.nasa.gov/mission_pages/cassini/multimedia/pia08329.html

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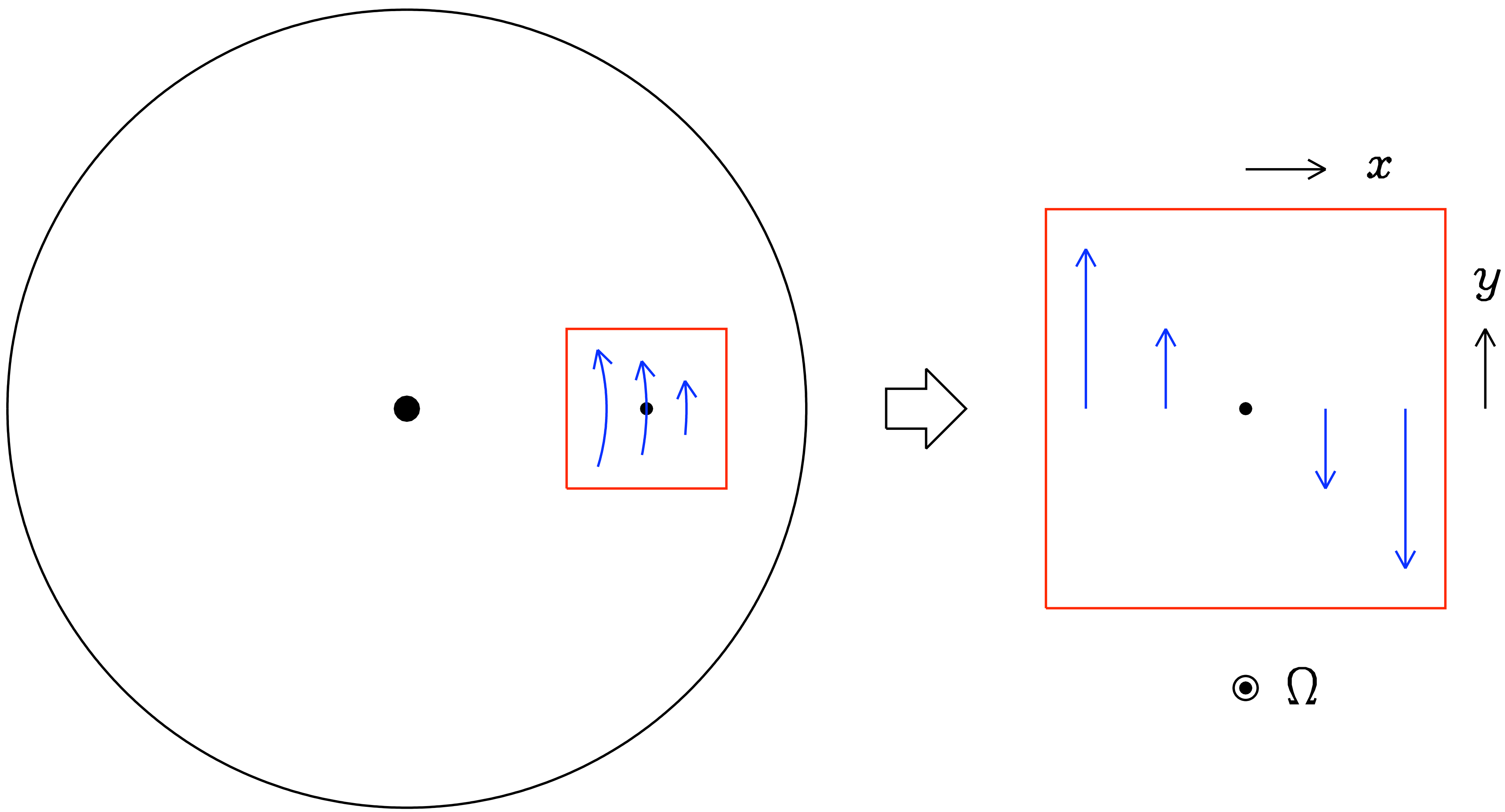
Local approximation

- Difficulties with global treatments of thin astrophysical discs:
- Computational:
  - Very large range of lengthscales and timescales (worse for thinner discs)
  - Inner and outer boundary conditions
- Analytical:
  - Cylindrical geometry and spatial inhomogeneity
  - Boundary conditions

Local approximation

- Shearing sheet / local approximation
  (Goldreich & Lynden-Bell 1965)
- Local model of a differentially rotating disc



Local approximation

- Consider an orbiting reference point with cylindrical coordinates

$$(r, \phi, z) = (r_0, \phi_0 + \Omega_0 t, 0) \qquad \Omega_0 = \Omega(r_0)$$

- Use as origin of a local Cartesian coordinate system (x, y, z)

$$x = r - r_0 \qquad \text{radial}$$

$$y = r_0(\phi - \phi_0 - \Omega_0 t) \qquad \text{azimuthal}$$

$$z = z \qquad \text{vertical}$$

- Orbital motion appears locally as a uniform rotation $\boldsymbol{\Omega}_0 = \Omega_0\, \boldsymbol{e}_z$ plus a linear shear flow $\boldsymbol{u}_0 = -S_0 x\, \boldsymbol{e}_y$

$$S = -r\frac{\mathrm{d}\Omega}{\mathrm{d}r} \qquad \text{rate of orbital shear}$$

Local approximation

- Effective potential in rotating frame
  (different from previous effective potential under $h = \mathrm{cst}$)
  expanded to second order in x and z

$$= \Phi(r, z) - \frac{1}{2}\Omega_0^2 r^2$$

$$= \Phi(r_0, 0) + \cancel{\Phi_{,r}(r_0, 0)x} + \frac{1}{2}\Phi_{,rr}(r_0, 0)x^2 + \frac{1}{2}\Phi_{,zz}(r_0, 0)z^2$$

$$-\frac{1}{2}\Omega_0^2(r_0^2 + \cancel{2r_0 x} + x^2)$$

$$= \cancel{\mathrm{cst}} + \frac{1}{2}[\partial_r(r\Omega^2) - \Omega^2]_0 x^2 + \frac{1}{2}\Omega_{z0}^2 z^2$$

$$= -\Omega_0 S_0 x^2 + \frac{1}{2}\Omega_{z0}^2 z^2$$

Local approximation

- Particle dynamics in local approximation

$$\ddot{x} - 2\Omega_0 \dot{y} = 2\Omega_0 S_0 x$$
$$\ddot{y} + 2\Omega_0 \dot{x} = 0$$
$$\ddot{z} = -\Omega_{z0}^2 z$$

(Keplerian case
$S_0 = \frac{3}{2}\Omega_0$
$\Omega_{z0} = \Omega_0$
→"Hill's equations")
(without satellite)

- Simple orbital motion:

$$x = \mathrm{cst}$$
$$\dot{y} = -S_0 x$$

Coriolis force balances effective potential gradient

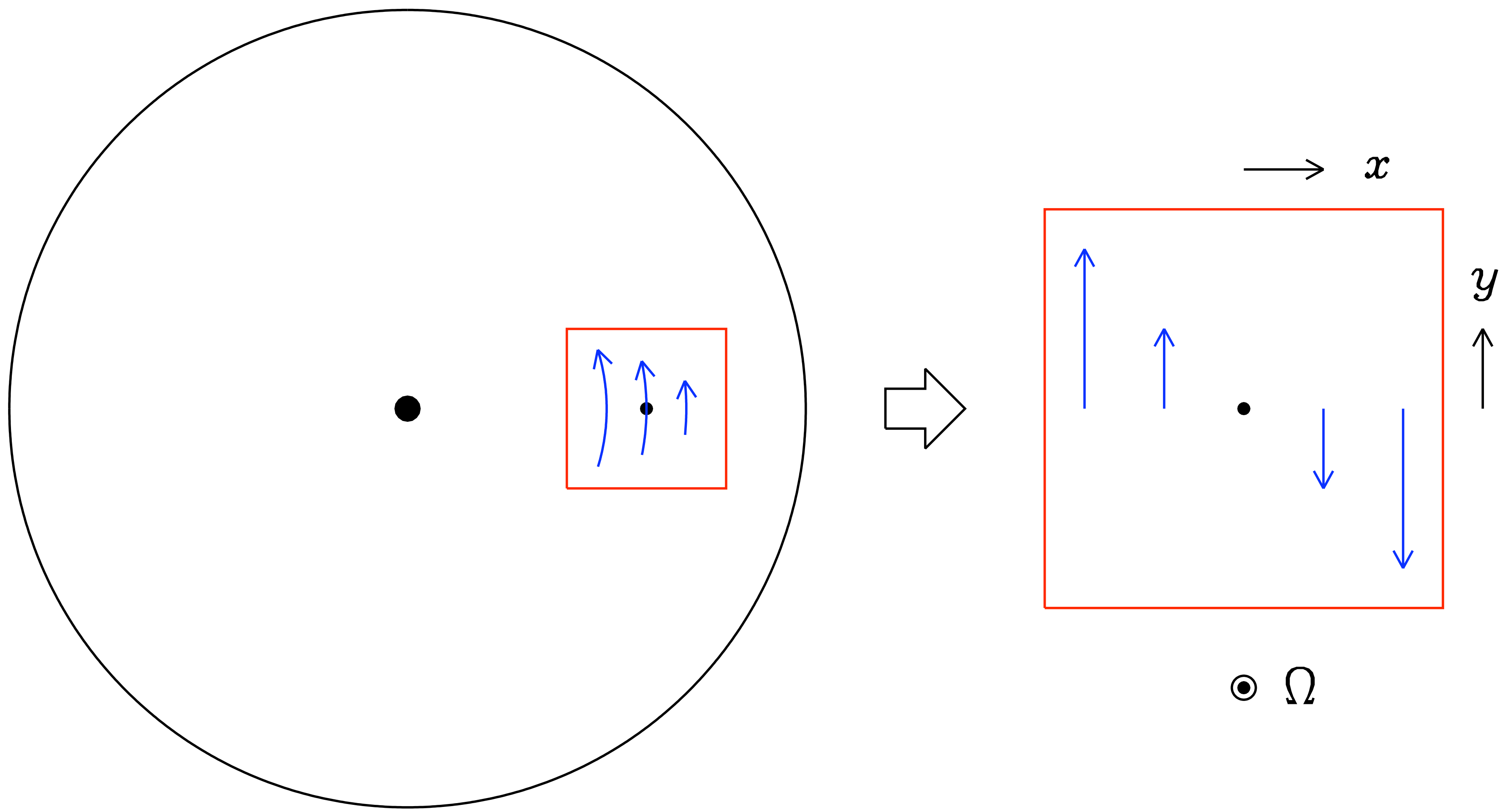
- General solution involves horizontal and vertical oscillations
- Canonical y momentum (per unit mass):

$$p_y = \dot{y} + 2\Omega_0 x = \mathrm{cst}$$

  - Plays role of specific angular momentum in local approximation
  - Has uniform gradient in simple orbital motion: $p_y = (2\Omega_0 - S_0)x$

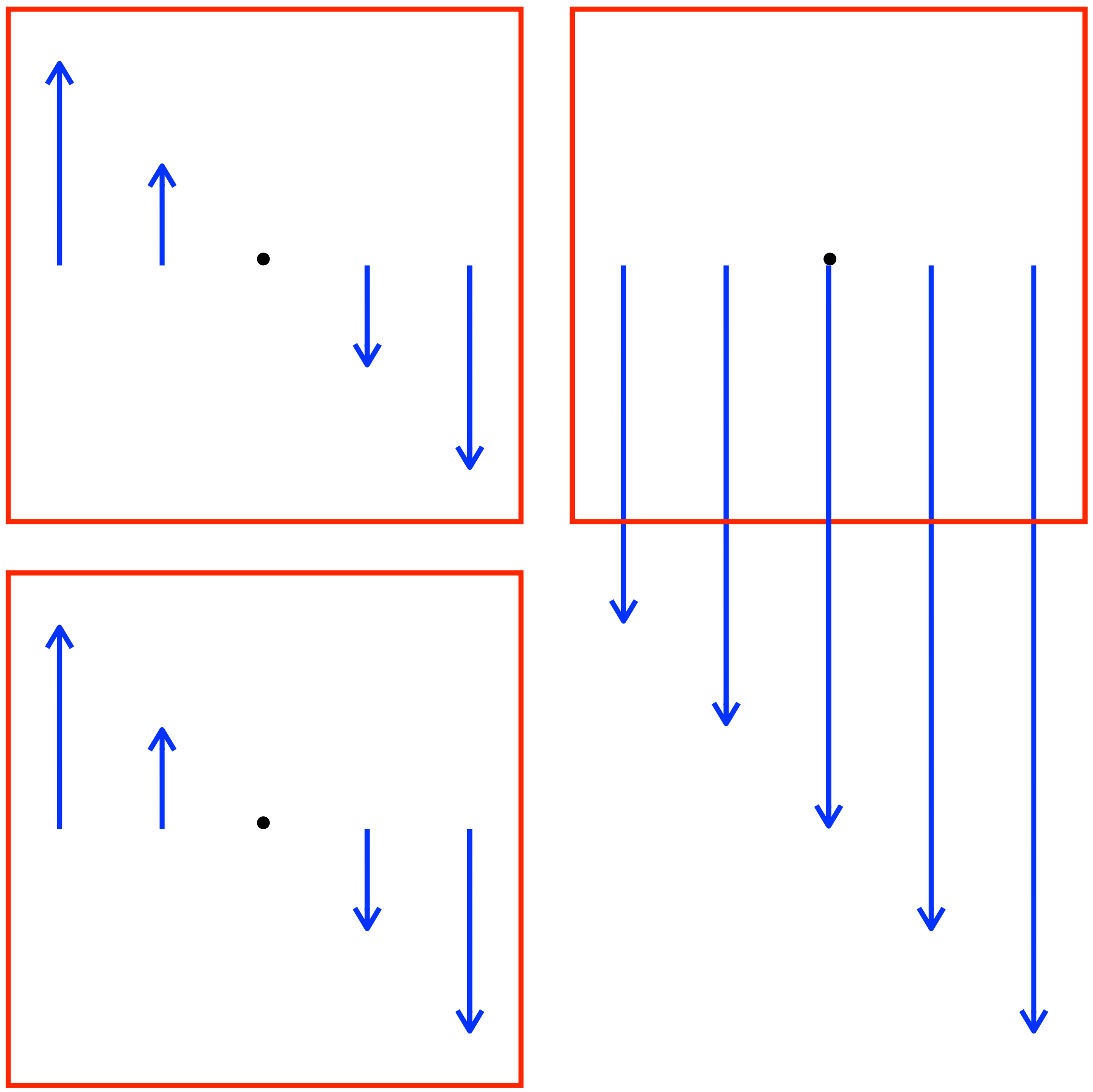
Local approximation

- Symmetries of local approximation:
  (higher than those of original disc!)
  - Spatial homogeneity (horizontally):
    every point in xy plane is equivalent (up to Galiliean boost)
  - Rotation by π about z axis



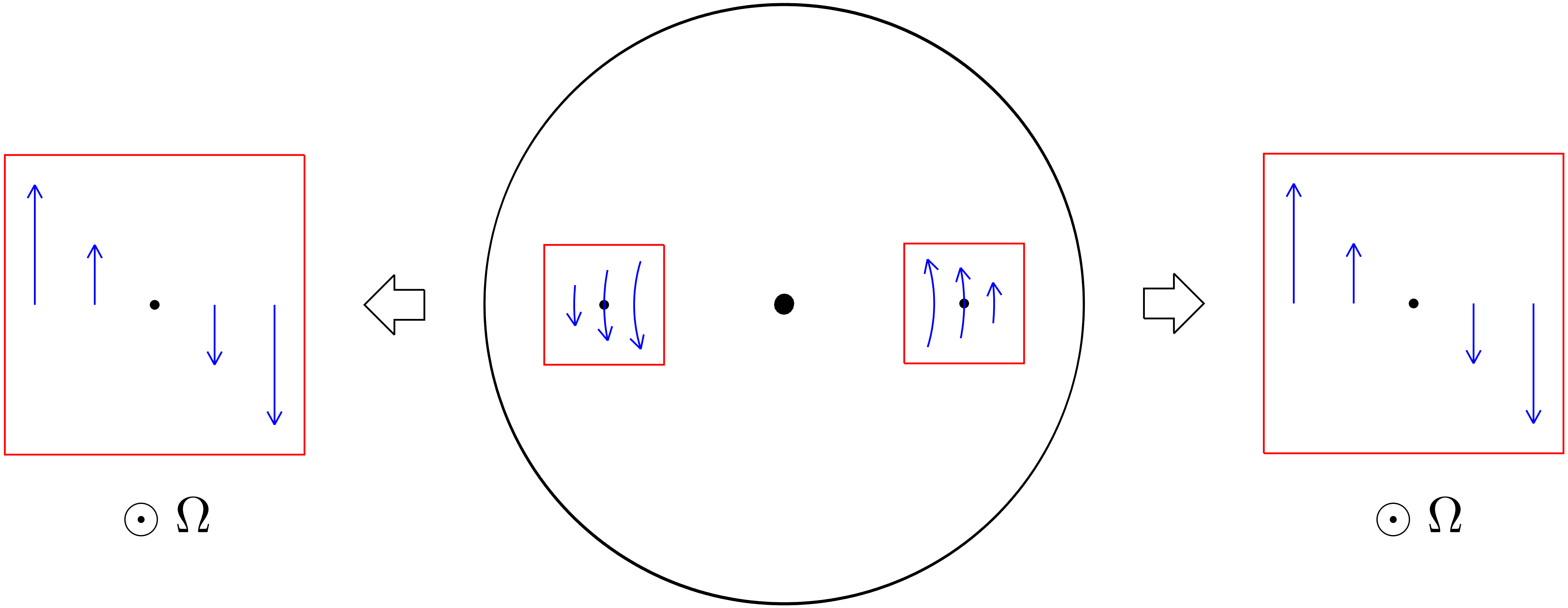
Local approximation

Spatial homogeneity



Local approximation

Rotational symmetry

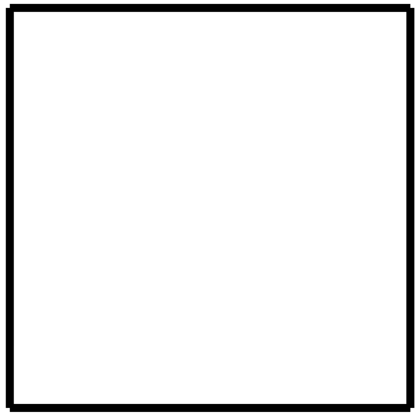


- Direction to central object cannot be determined
- No accretion flow therefore expected
- Local model knows about Ω (and S) but not about r

Local approximation

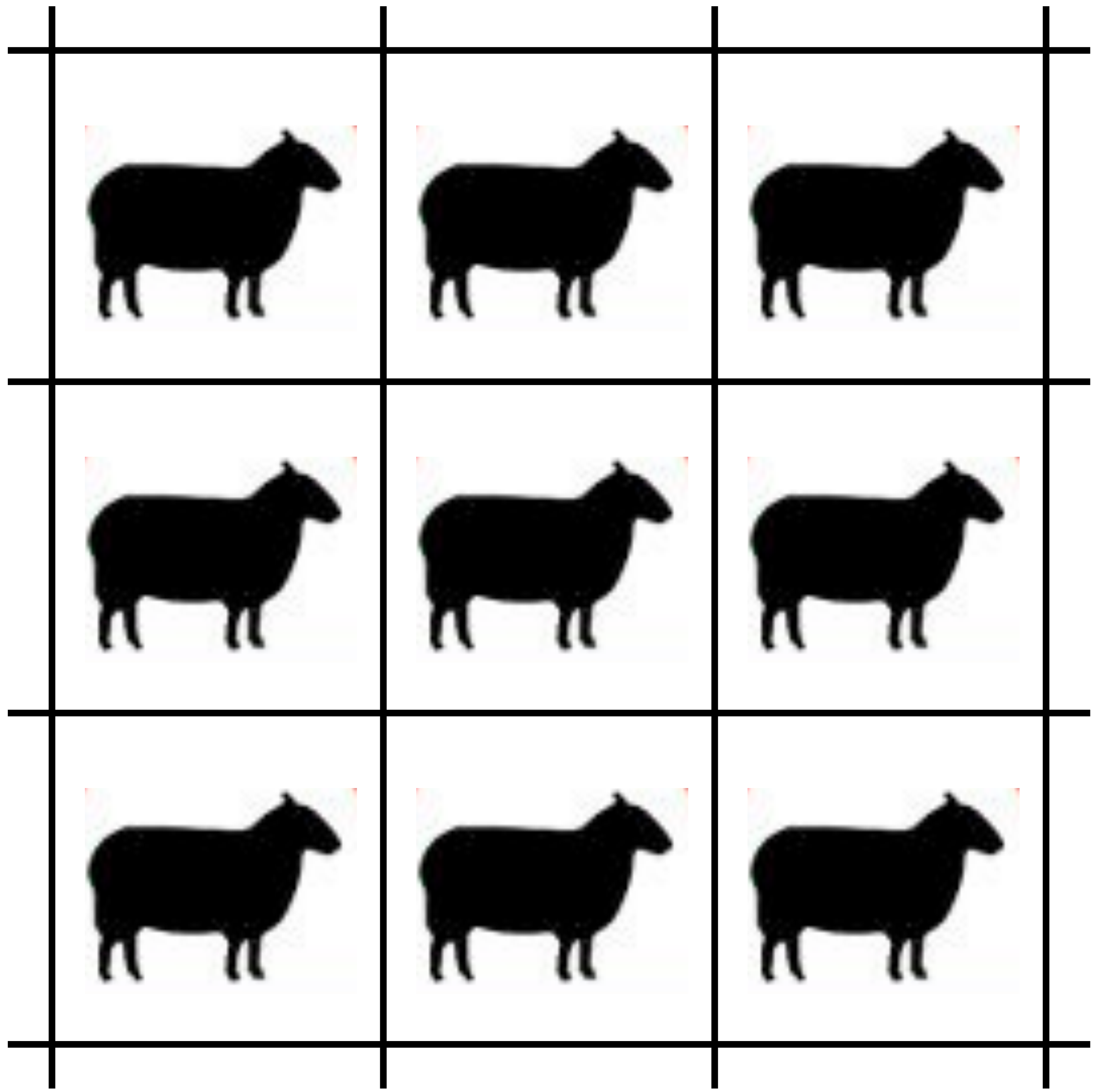
- Boundary conditions of shearing sheet
  - Horizontally unbounded or apply (modified) periodic boundary conditions
  - Vertical structure:
    - Ignore z completely (2D shearing sheet)
    - Neglect vertical gravity: homogeneous in z
    - Include vertical gravity: isothermal, radiative, etc. models

Shearing box



Shearing box

Shearing box



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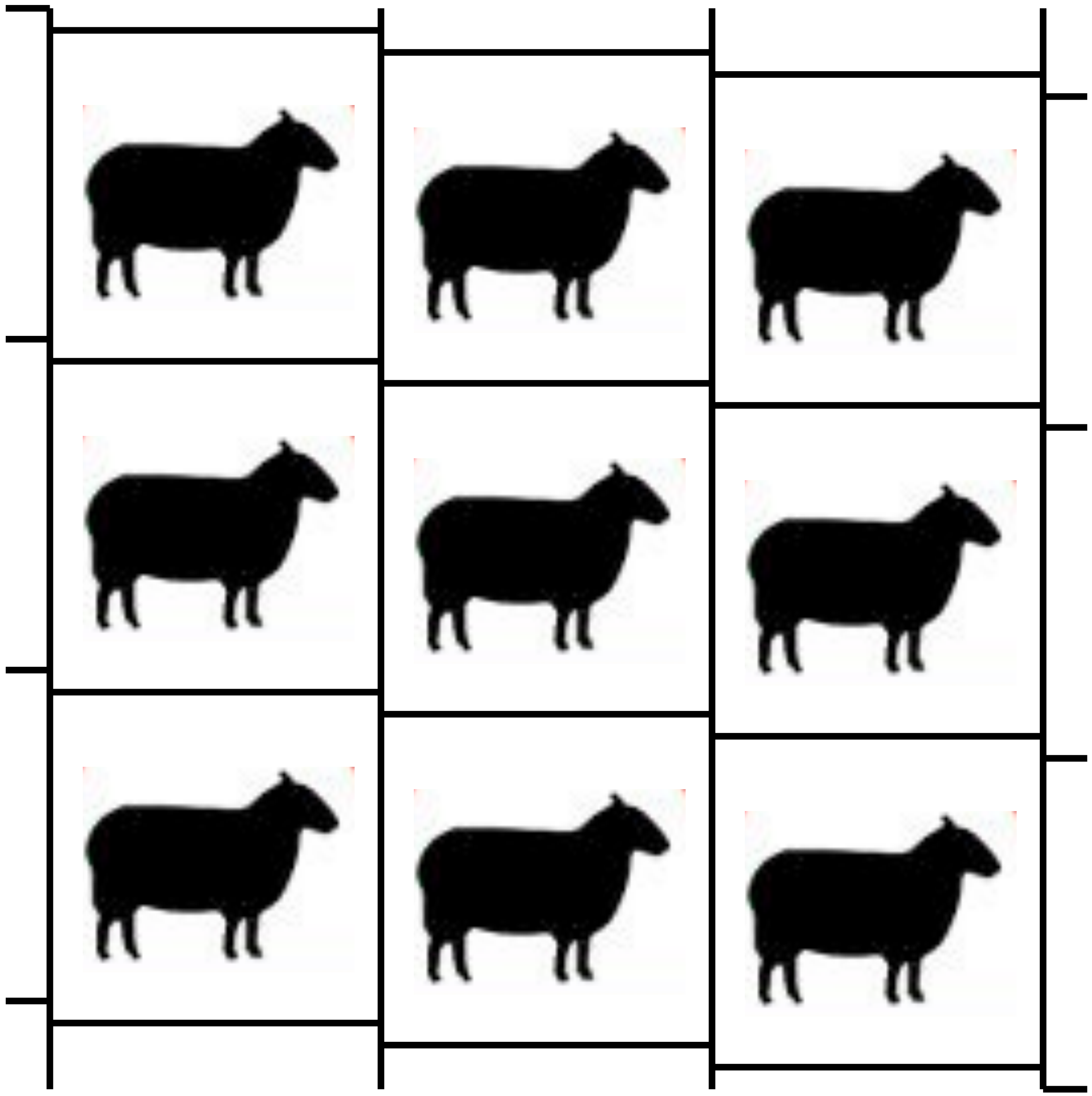

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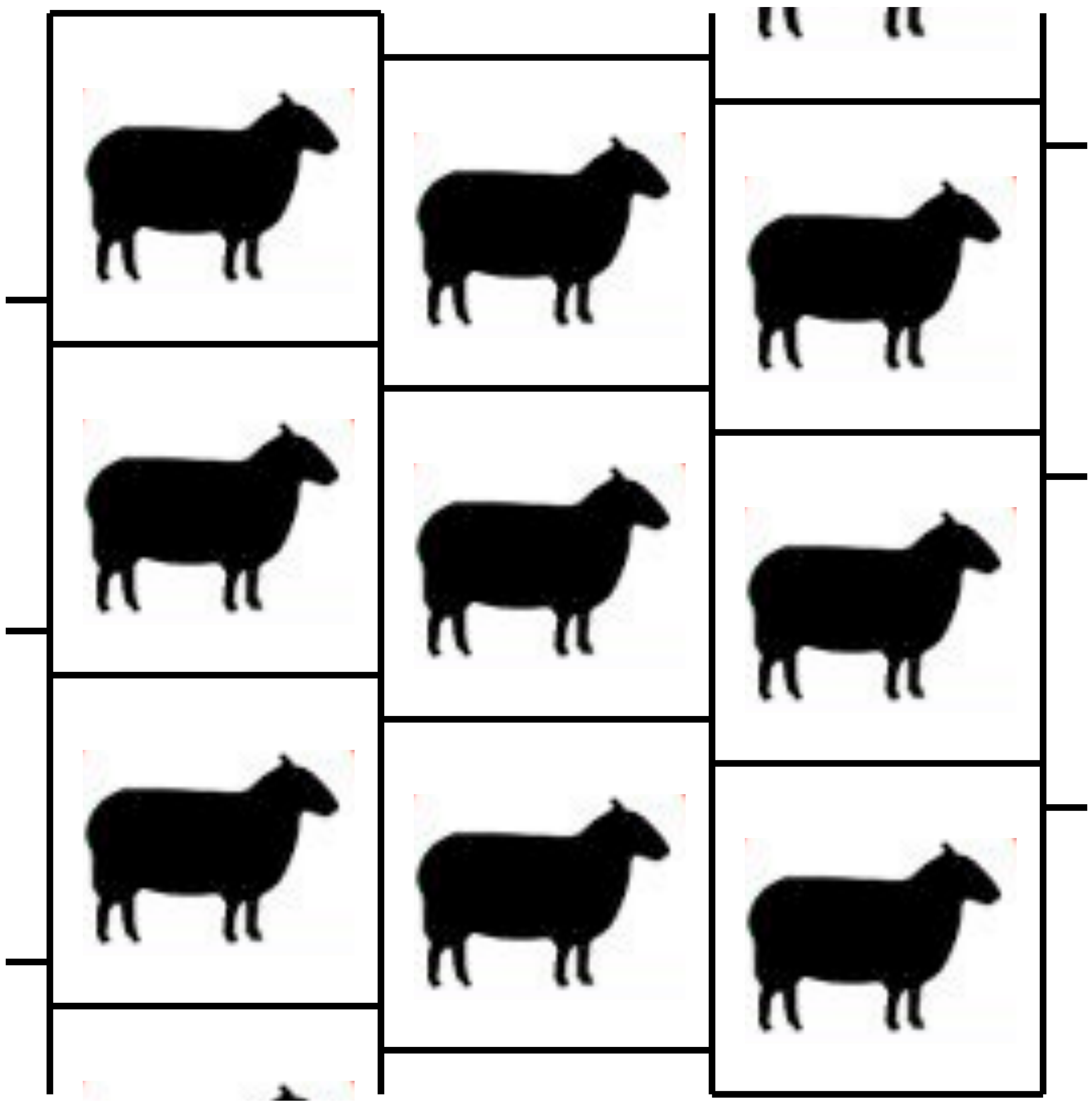
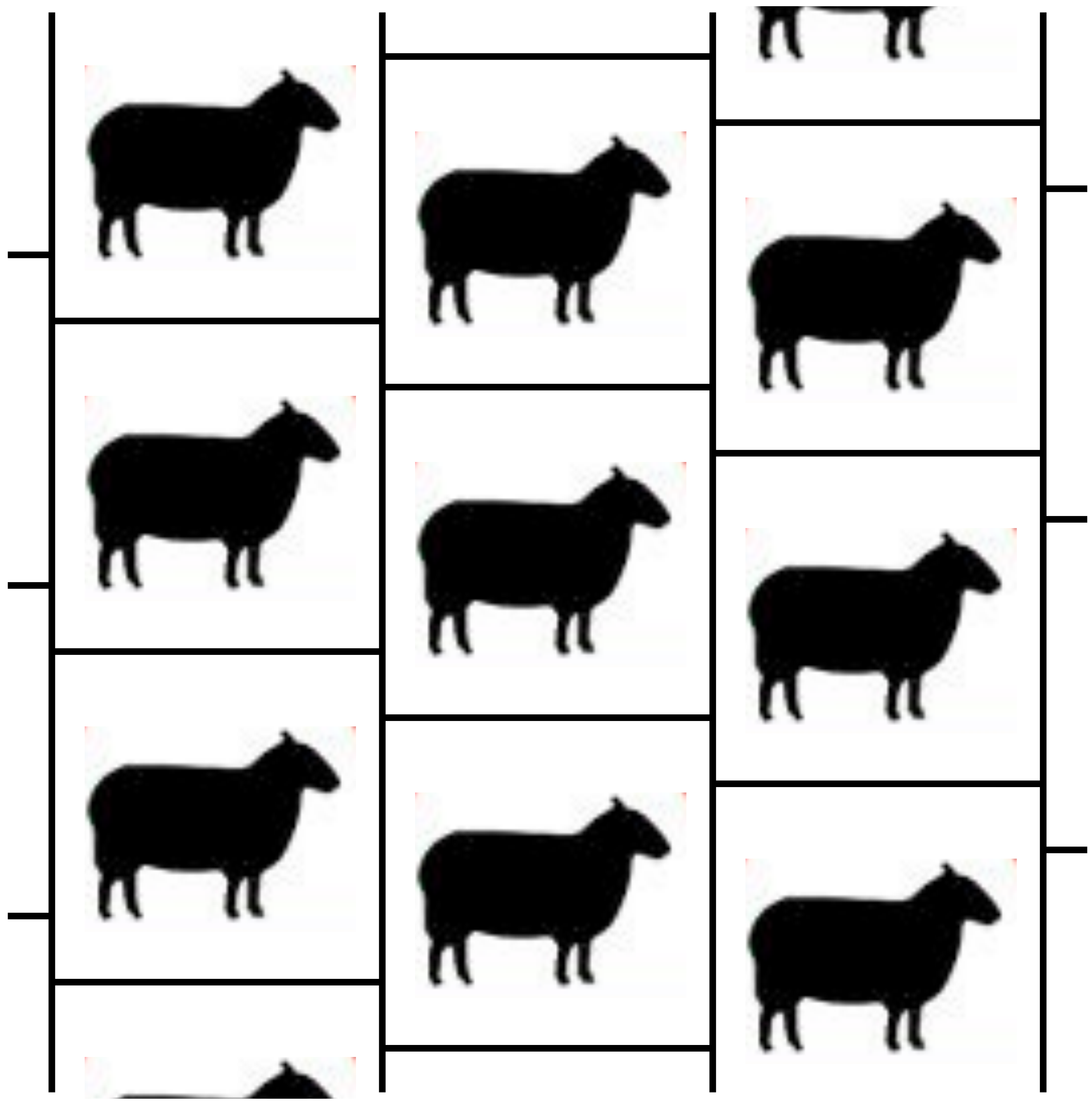
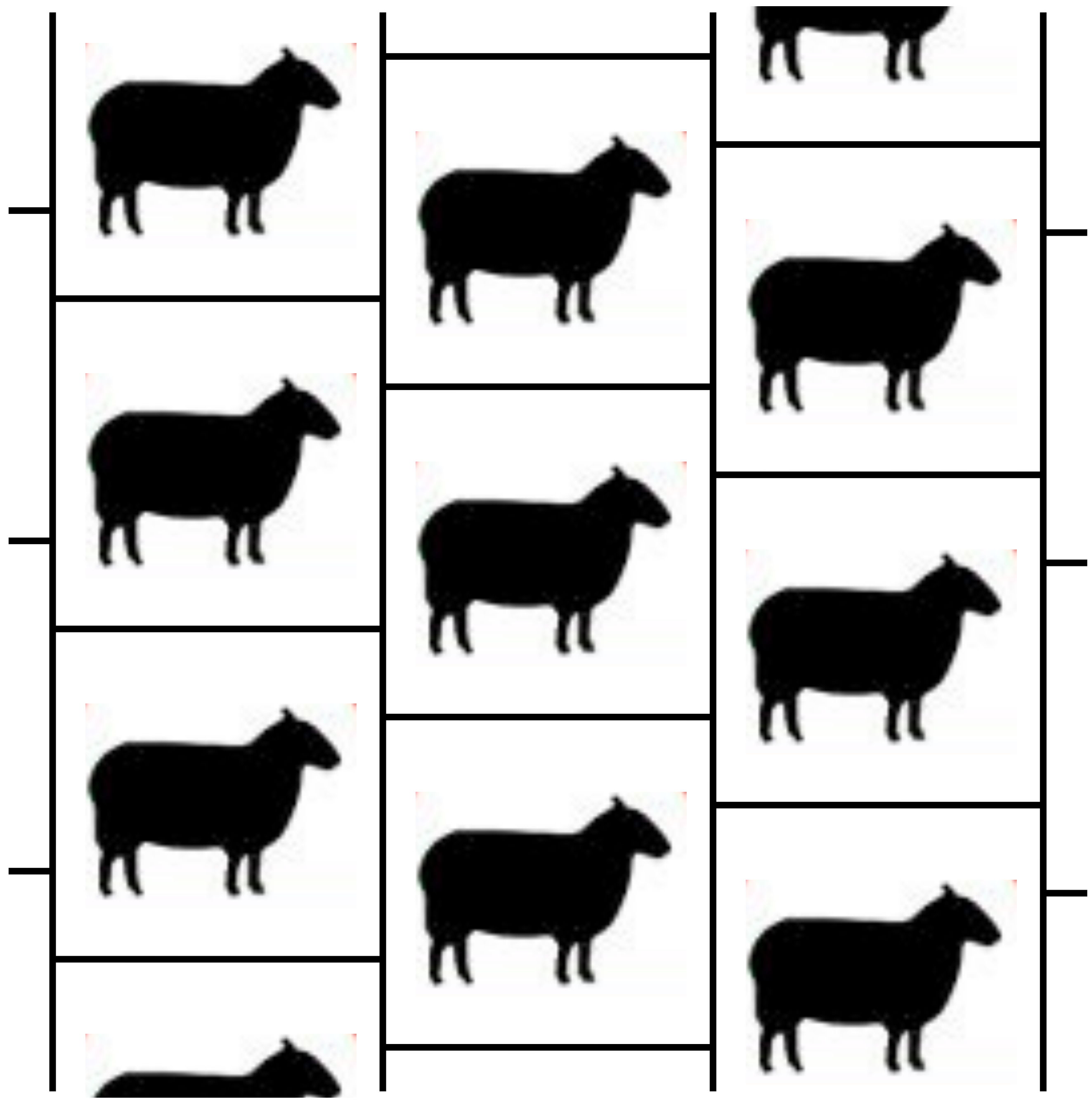


Shearing box



Shearing box



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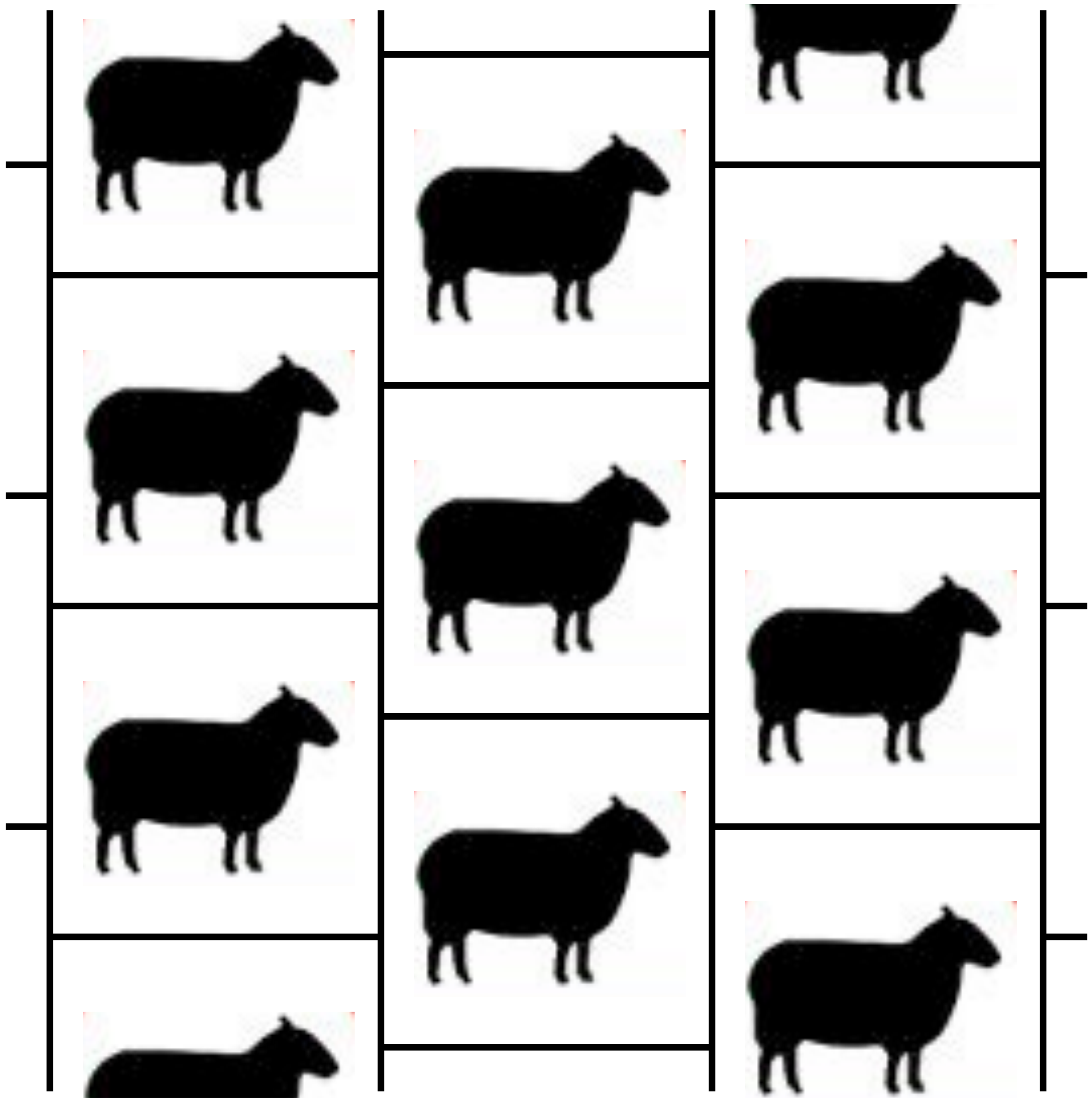

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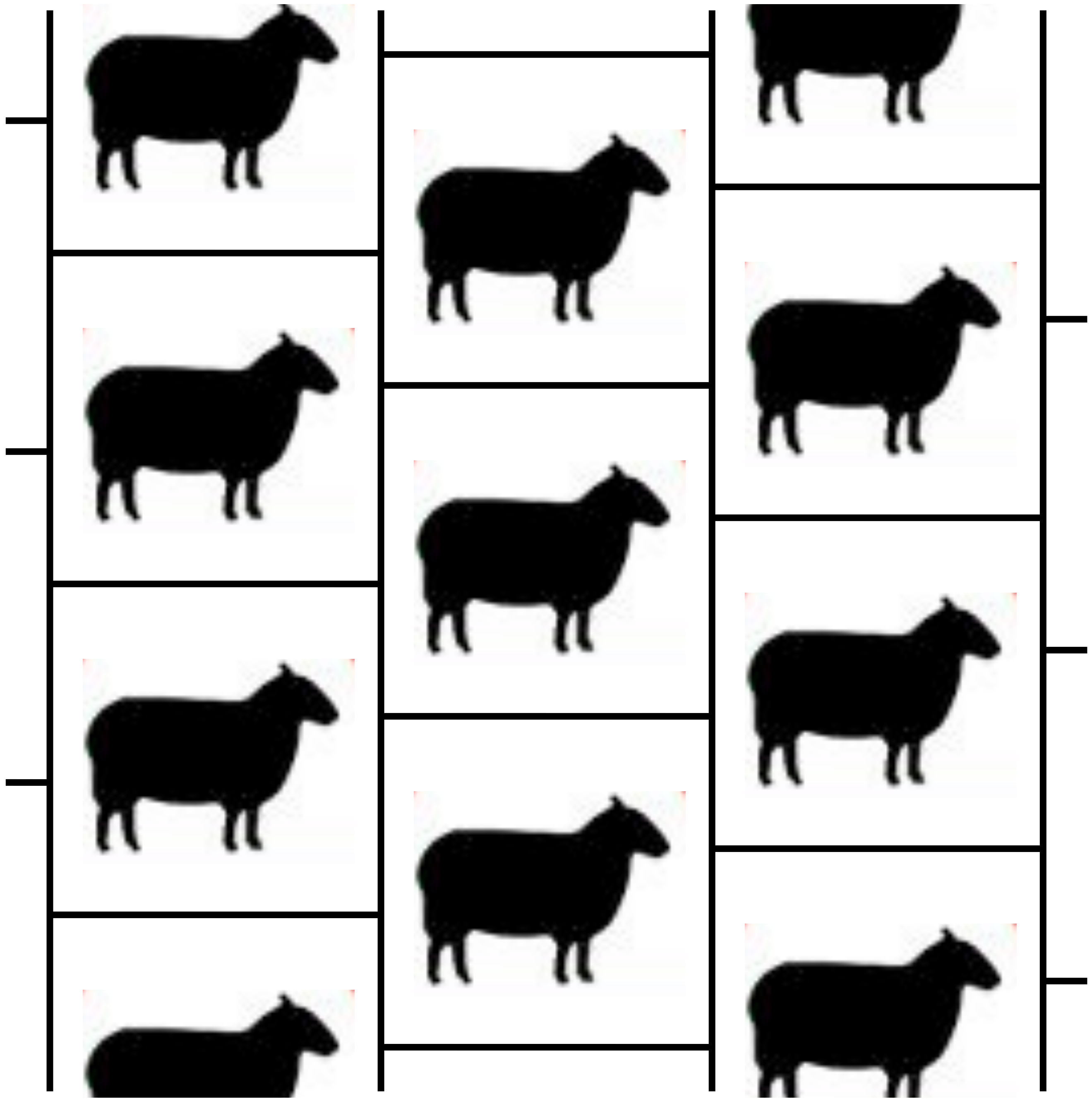

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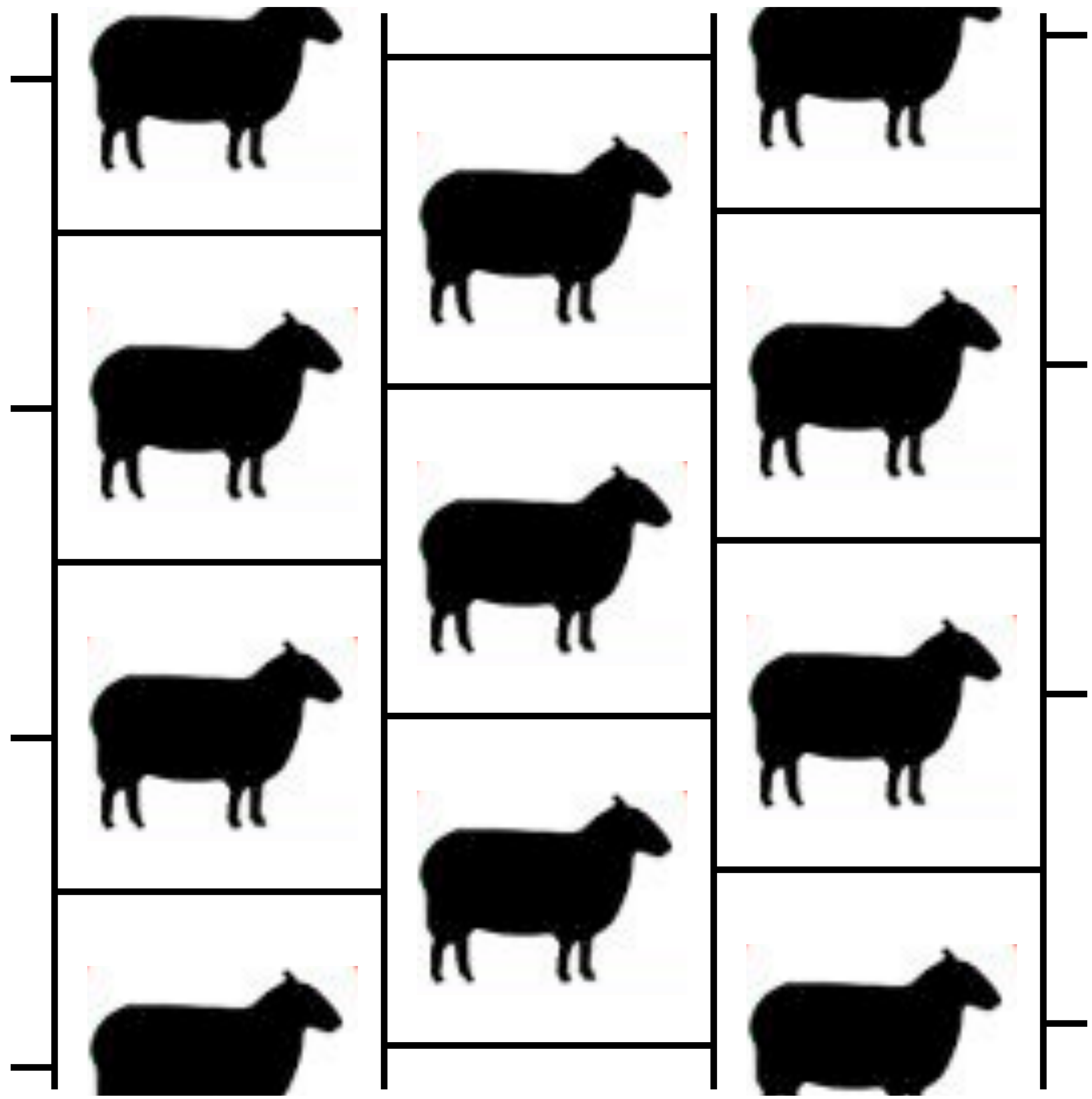

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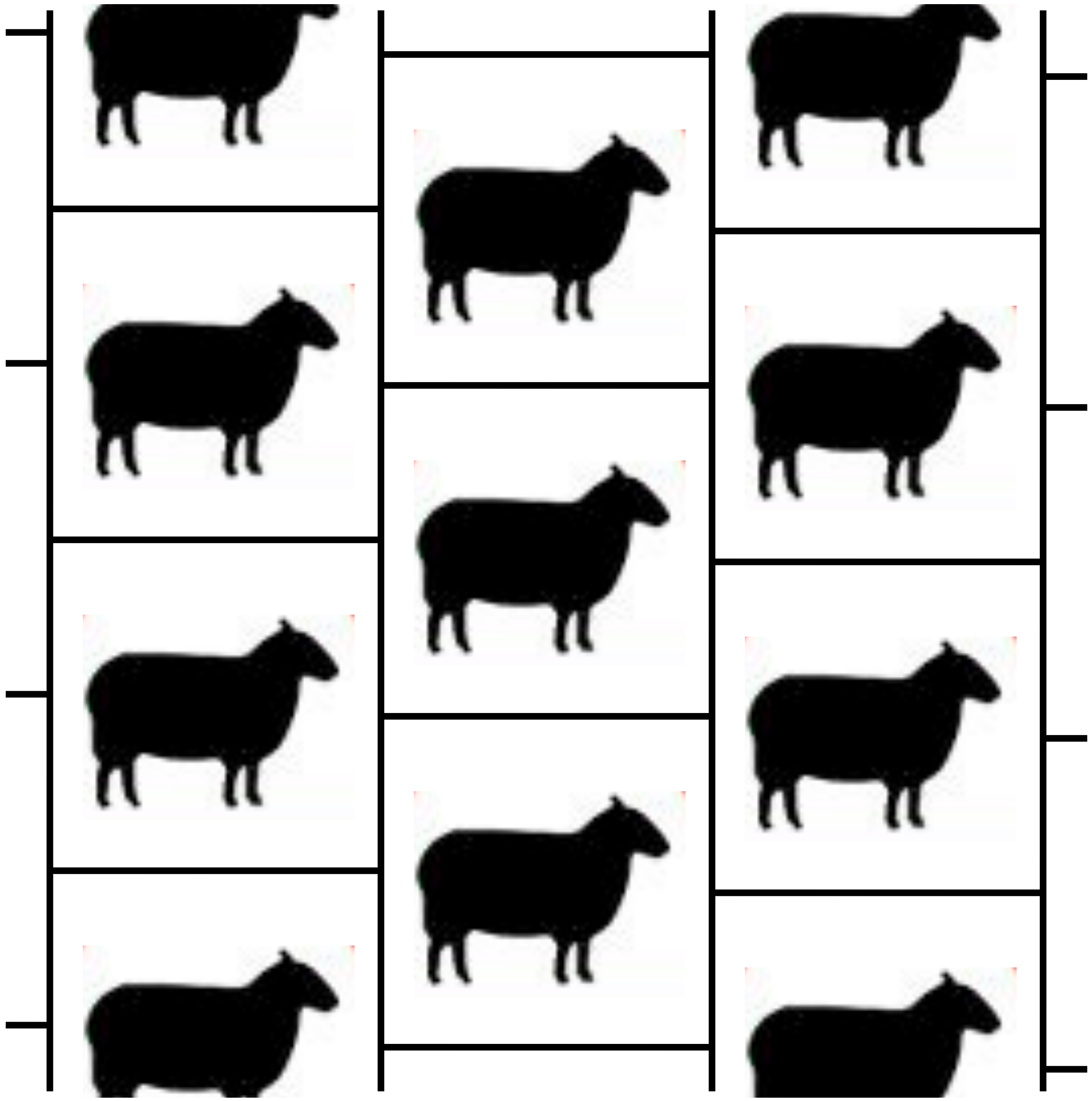

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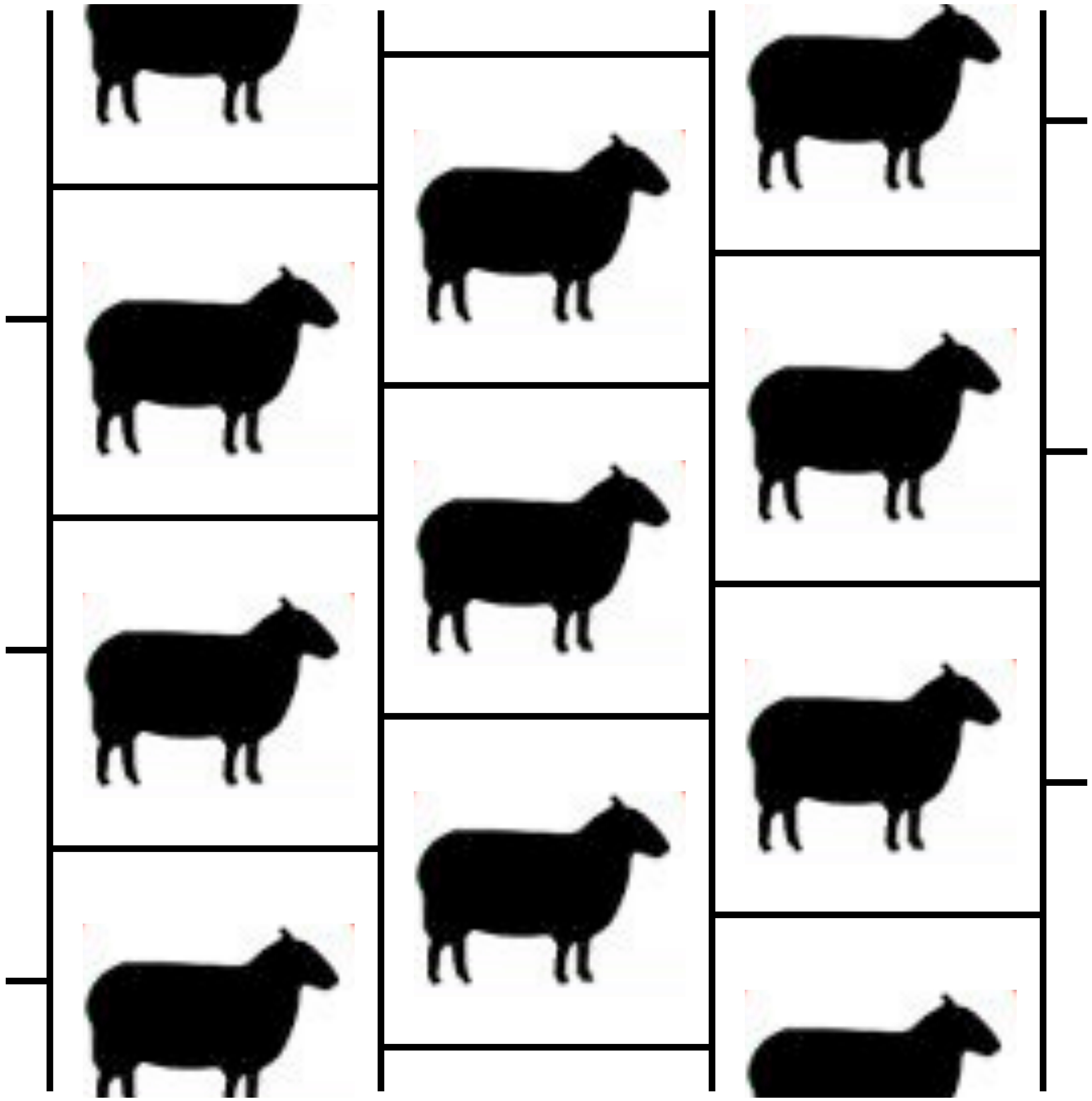

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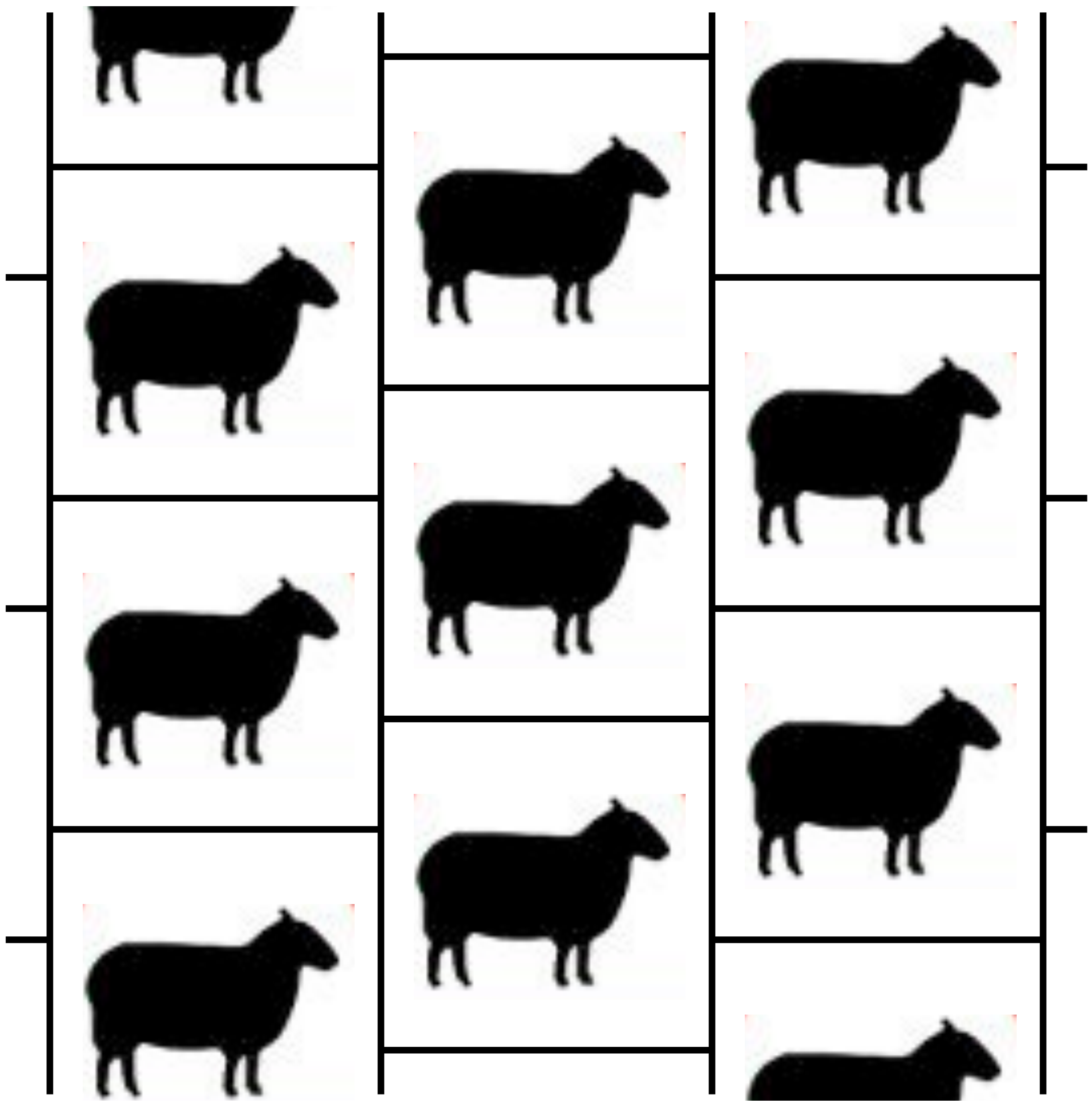

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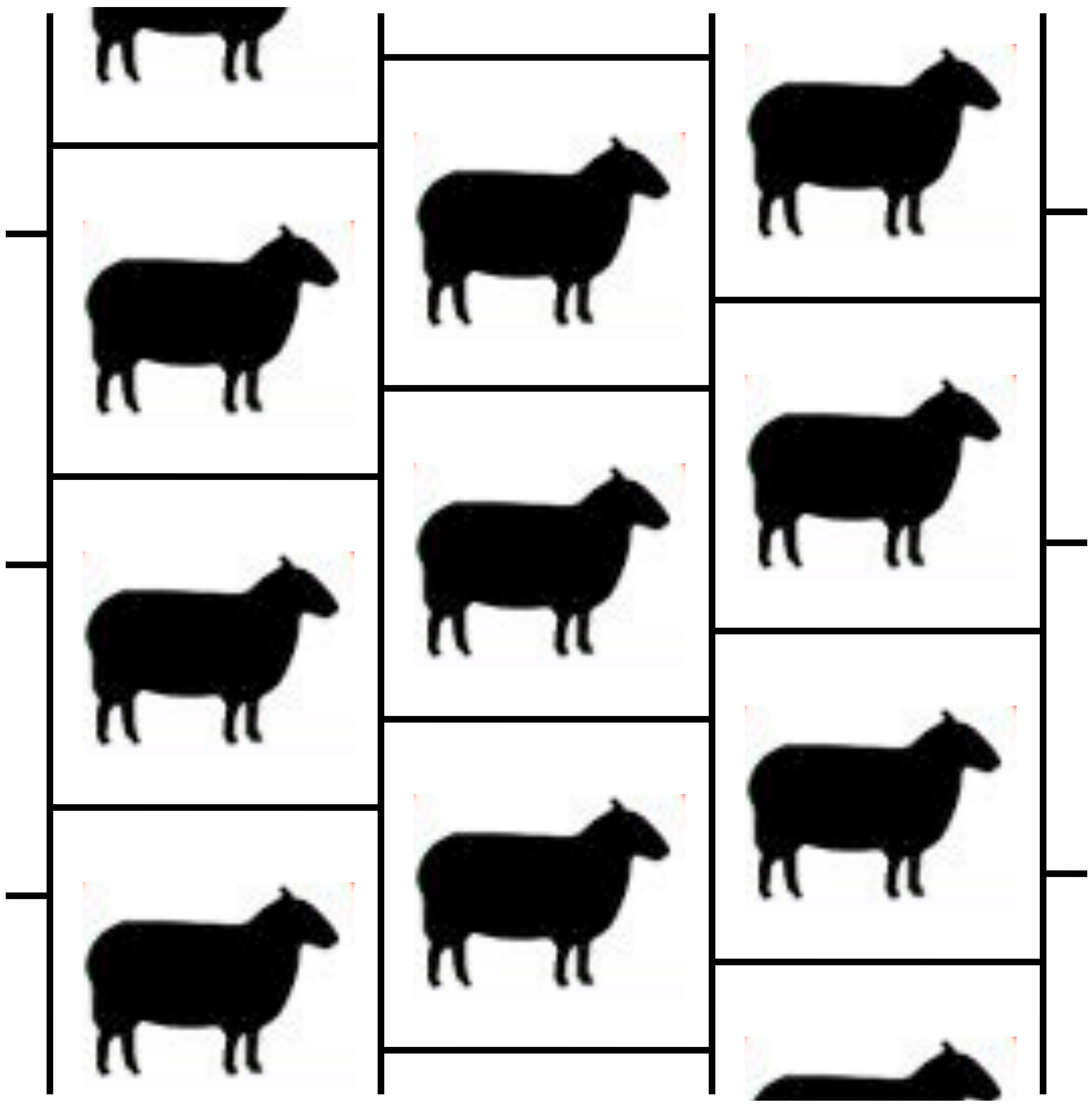

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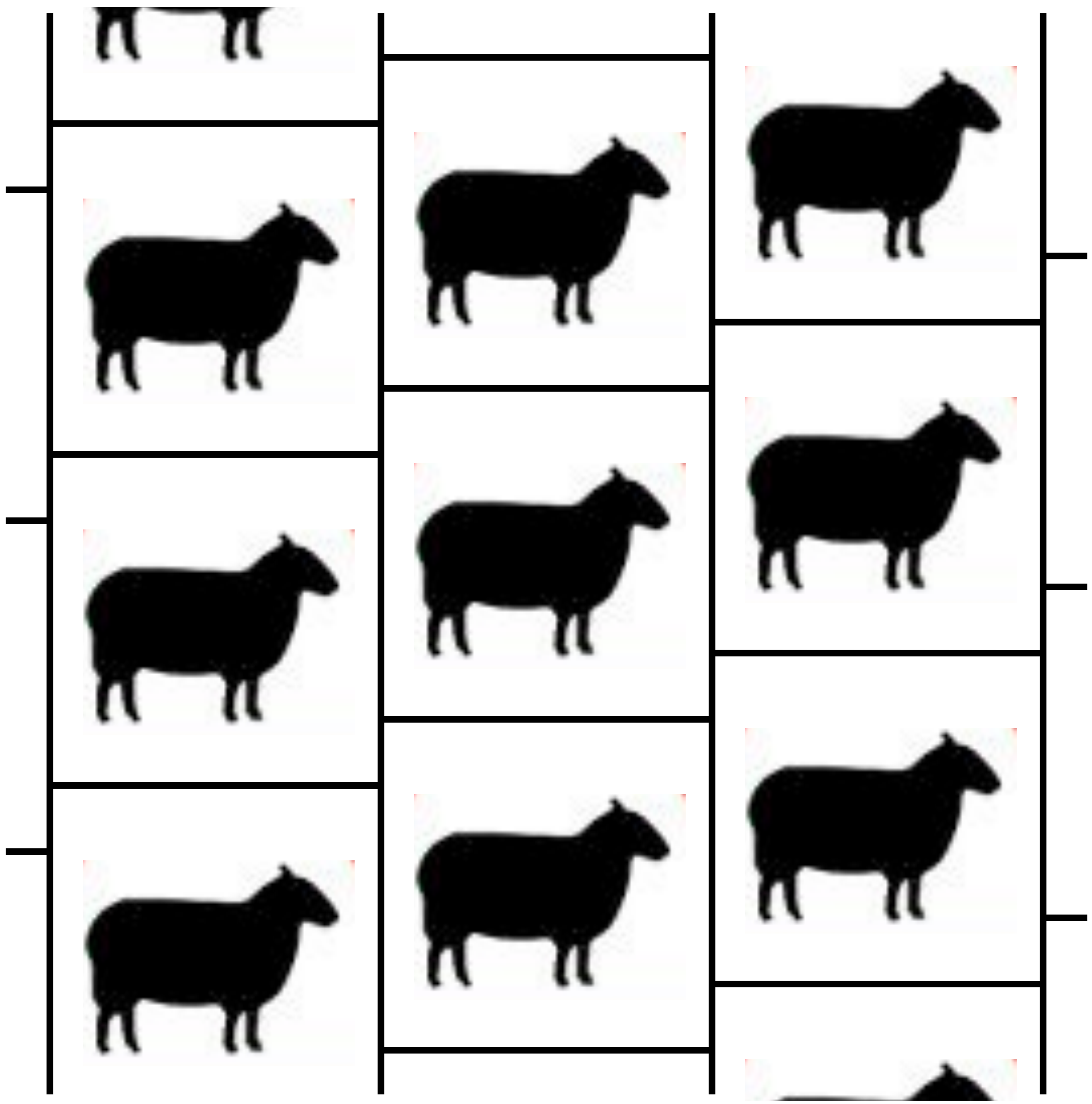

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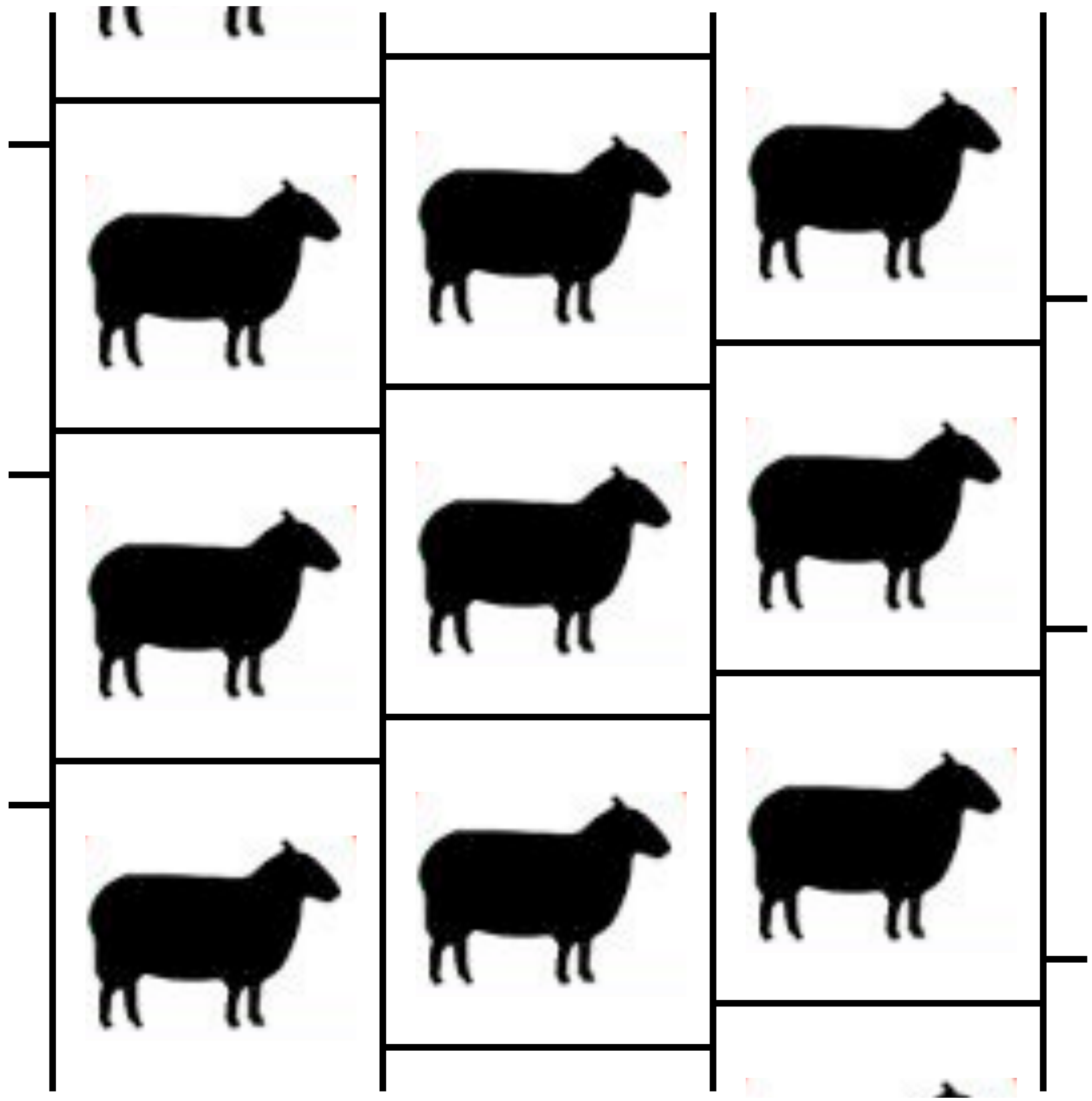

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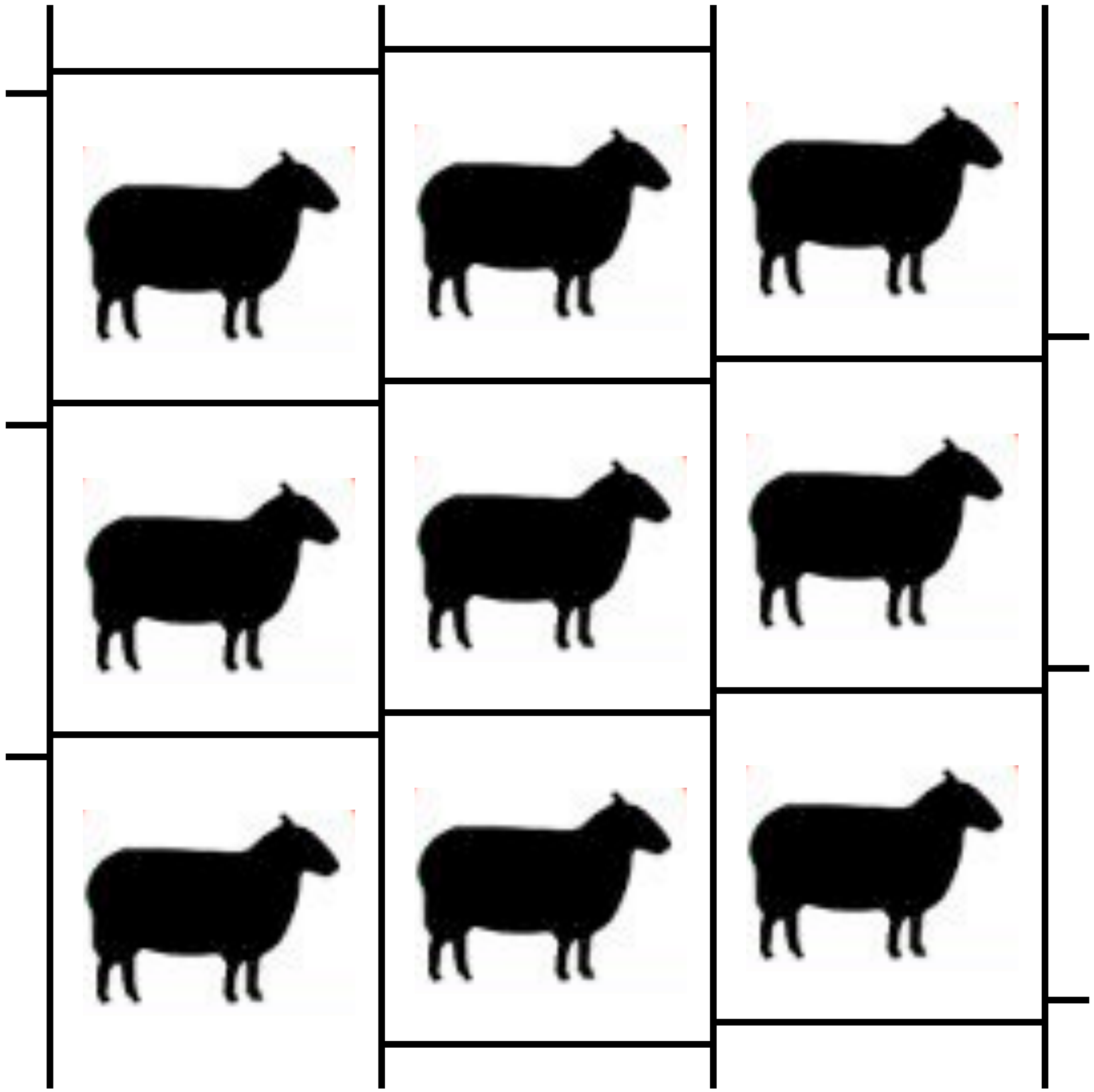

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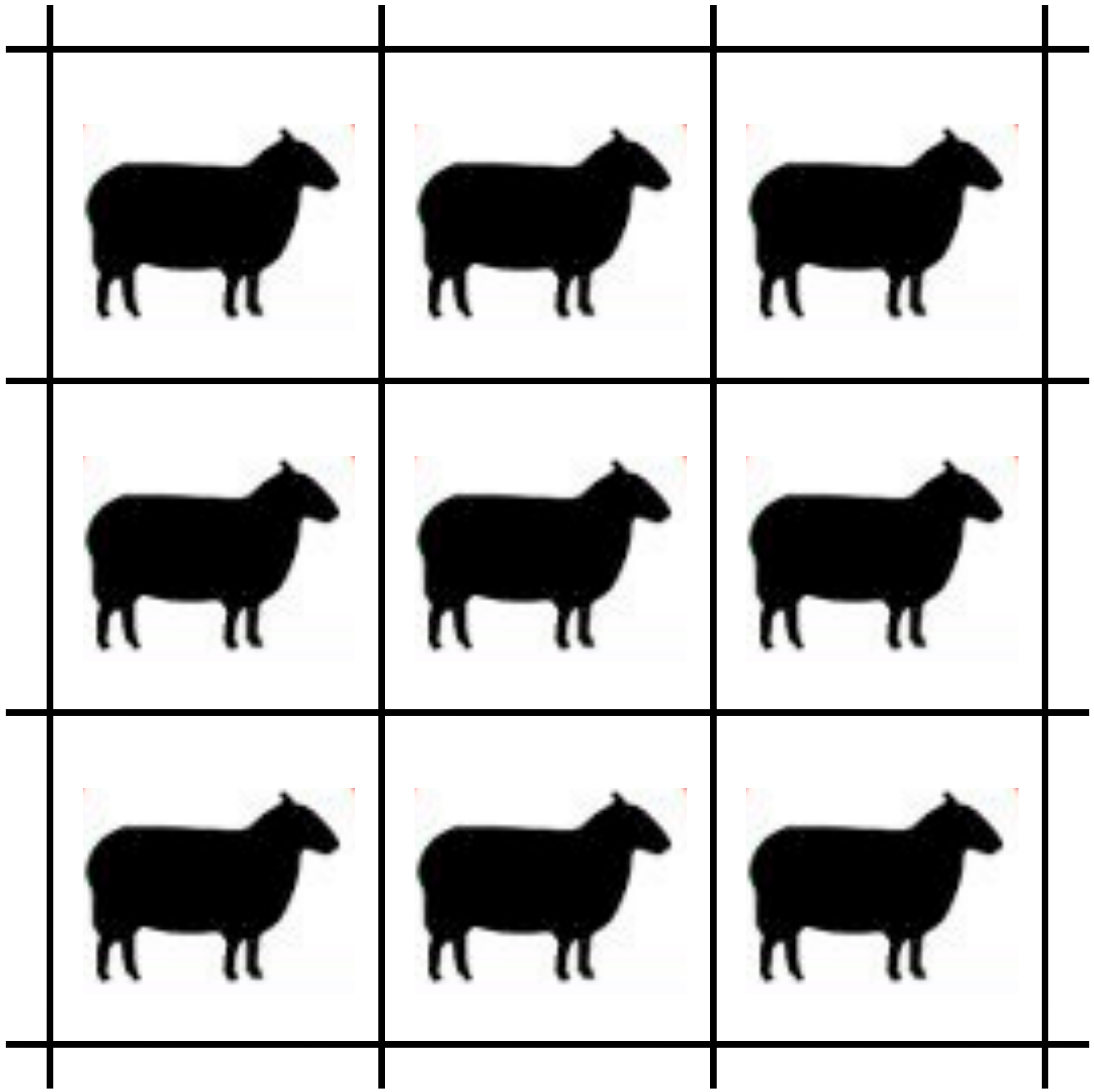


Incompressible fluid dynamics in shearing sheet

- Homogeneous incompressible fluid
- 3D system, unbounded or periodic in x, y, z
- Uniform kinematic viscosity ν [discuss its role]

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega}_0 \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

neglect (balanced
by pressure gradient)

- Effective potential $\Phi = -\Omega_0 S_0 x^2 + \cancel{\tfrac{1}{2}\Omega_{z0}^2 z^2}$
- Basic state:

$$\boldsymbol{u} = \boldsymbol{u}_0 = -S_0 x \, \boldsymbol{e}_y$$

$$p = p_0 = \mathrm{cst}$$

- Uniform viscous stress, but no divergence and so no accretion flow

Incompressible fluid dynamics in shearing sheet

- Perturbations (not necessarily small):

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{v}(x, y, z, t)$$

$$p = p_0 + p'(x, y, z, t)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{u}_0 + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} + 2\boldsymbol{\Omega}_0 \times \boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p' + \nu\nabla^2\boldsymbol{v}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

- Now drop the subscript 0 on Ω_0 and S_0 and let $\psi = p'/\rho$:

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\right) v_x - 2\Omega v_y = -\frac{\partial \psi}{\partial x} + \nu\nabla^2 v_x$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\right) v_y + (2\Omega - S) v_x = -\frac{\partial \psi}{\partial y} + \nu\nabla^2 v_y$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\right) v_z = -\frac{\partial \psi}{\partial z} + \nu\nabla^2 v_z$$

Incompressible fluid dynamics in shearing sheet

- Shearing waves (after Kelvin / Thomson 1887):
- Consider a plane-wave disturbance of the form

$$\boldsymbol{v}(\boldsymbol{x},t) = \mathrm{Re}\left\{\tilde{\boldsymbol{v}}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$$
$$\psi(\boldsymbol{x},t) = \mathrm{Re}\left\{\tilde{\psi}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$$

time-dependent wavevector

- Then

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\boldsymbol{v} = \mathrm{Re}\left\{\left[\frac{\mathrm{d}\tilde{\boldsymbol{v}}}{\mathrm{d}t} + \left(\mathrm{i}\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}\cdot\boldsymbol{x} - Sx\,\mathrm{i}k_y\right)\tilde{\boldsymbol{v}}\right]\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$$

- If we choose

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = Sk_y\,\boldsymbol{e}_x$$

then two terms cancel and we are left with $\dfrac{\mathrm{d}\tilde{\boldsymbol{v}}}{\mathrm{d}t}$

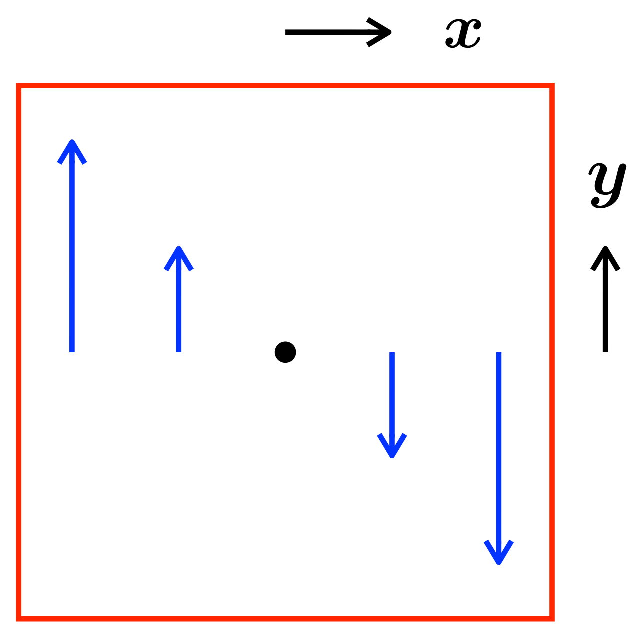
- This means

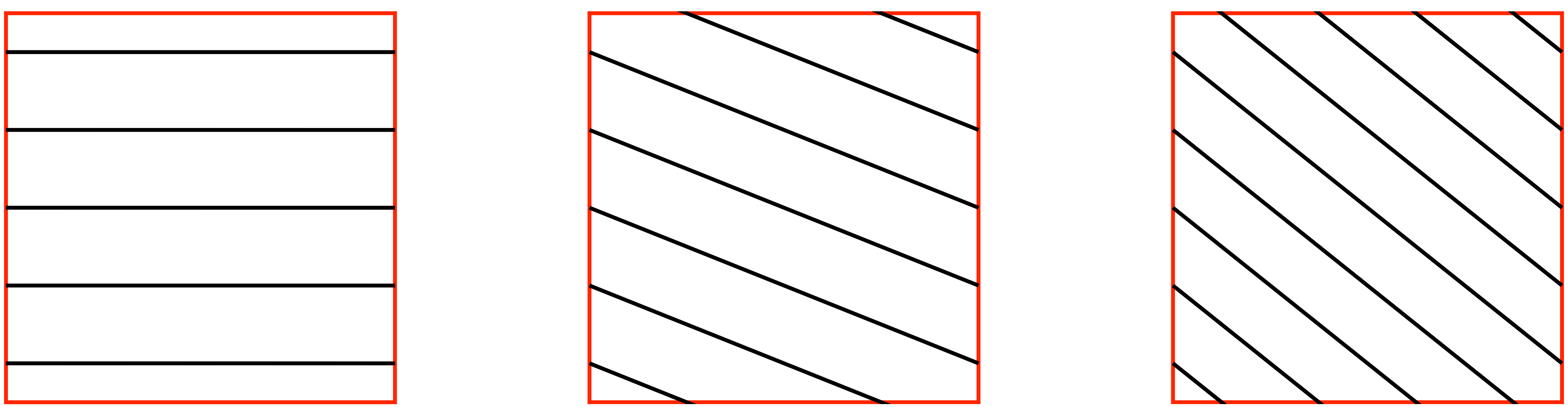
$$k_x = k_{x0} + Sk_yt \qquad k_y = \mathrm{cst} \qquad k_z = \mathrm{cst}$$

Incompressible fluid dynamics in shearing sheet

$$k_x = k_{x0} + S k_y t \qquad k_y = \mathrm{cst} \qquad k_z = \mathrm{cst}$$

- Tilting / shearing of wavefronts:

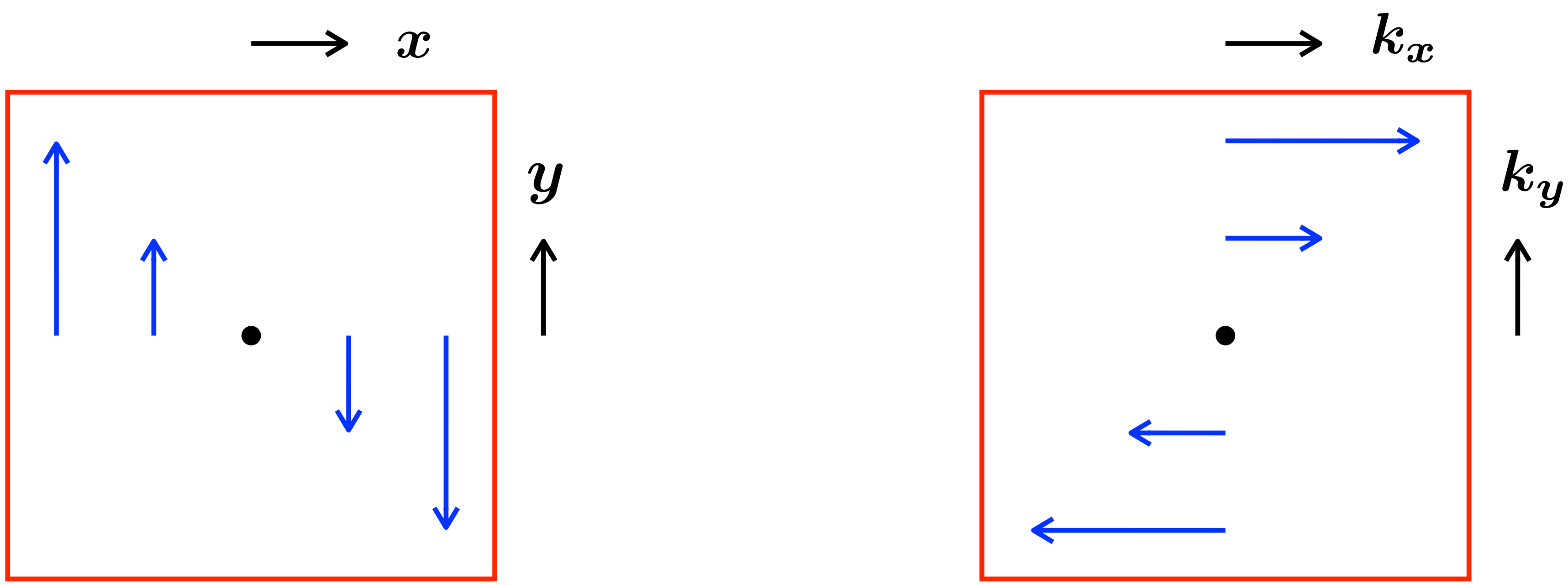




Incompressible fluid dynamics in shearing sheet

$$k_x = k_{x0} + S k_y t \qquad k_y = \mathrm{cst} \qquad k_z = \mathrm{cst}$$

- Dual shear flow in Fourier space:



Incompressible fluid dynamics in shearing sheet

- Furthermore

$$\begin{aligned}
\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} &= \mathrm{Re}\left[\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\cdot\boldsymbol{\nabla}\,\mathrm{Re}\left[\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right] \\
&= \mathrm{Re}\left[\boldsymbol{k}\cdot\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right]\mathrm{Re}\left[\mathrm{i}\tilde{\boldsymbol{v}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right] \\
&= 0
\end{aligned}$$

because $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0 \quad \Rightarrow \quad \mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}}=0$

- Special result for incompressible fluid
- Nonlinearity doesn’t vanish for a superposition of shearing waves

Incompressible fluid dynamics in shearing sheet

- Amplitude equations for shearing waves:

$$\frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}t} - 2\Omega\tilde{v}_y = -\mathrm{i}k_x\tilde{\psi} - \nu k^2\tilde{v}_x$$

$$\frac{\mathrm{d}\tilde{v}_y}{\mathrm{d}t} + (2\Omega - S)\tilde{v}_x = -\mathrm{i}k_y\tilde{\psi} - \nu k^2\tilde{v}_y \qquad k^2 = |\boldsymbol{k}|^2$$

$$\frac{\mathrm{d}\tilde{v}_z}{\mathrm{d}t} = -\mathrm{i}k_z\tilde{\psi} - \nu k^2\tilde{v}_z$$

$$\mathrm{i}\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$$

- Viscous terms are taken care of by a viscous decay factor

$$E_\nu(t) = \exp\left(-\int \nu k^2\,\mathrm{d}t\right)$$

$$= \exp\left\{-\nu\left[(k_{x0}^2 + k_y^2 + k_z^2)t + Sk_{x0}k_yt^2 + \tfrac{1}{3}S^2k_y^2t^3\right]\right\}$$

- Faster than exponential decay if $k_y \neq 0$

Incompressible fluid dynamics in shearing sheet

- Write $\tilde{\boldsymbol{v}} = E_\nu(t)\hat{\boldsymbol{v}}(t),\ \tilde{\psi} = E_\nu(t)\hat{\psi}(t)$ to obtain inviscid problem

$$\begin{aligned}
\frac{\mathrm{d}\hat{v}_x}{\mathrm{d}t} - 2\Omega\hat{v}_y &= -\mathrm{i}k_x\hat{\psi} \\
\frac{\mathrm{d}\hat{v}_y}{\mathrm{d}t} + (2\Omega - S)\hat{v}_x &= -\mathrm{i}k_y\hat{\psi} \\
\frac{\mathrm{d}\hat{v}_z}{\mathrm{d}t} &= -\mathrm{i}k_z\hat{\psi}
\end{aligned}$$

$$\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{v}} = 0$$

- Eliminate variables in favour of $\hat{v}_x$ to obtain (after algebra)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k^2\hat{v}_x) + \kappa^2 k_z^2\hat{v}_x = 0$$

$$\kappa^2 = 2\Omega(2\Omega - S)$$

square of epicyclic frequency in local approximation

Incompressible fluid dynamics in shearing sheet

- Analysis of axisymmetric/unsheared waves $(k_y = 0)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k^2\hat{v}_x) + \kappa^2 k_z^2 \hat{v}_x = 0$$

- Constant coefficients, so exponential / sinusoidal solutions
- Inviscid case:
  - Oscillations (inertial waves) if $\kappa^2 > 0$
  - Exponential growth if $\kappa^2 < 0$
- With viscosity, include factor $E_\nu = \exp(-\nu k^2 t)$:
  - Damped oscillations if $\kappa^2 > 0$
  - Unstable to sufficiently long wavelengths if $\kappa^2 < 0$

Incompressible fluid dynamics in shearing sheet

- Analysis of non-axisymmetric/sheared waves $(k_y \neq 0)$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(k^2\hat{v}_x) + \kappa^2 k_z^2 \hat{v}_x = 0$$

- Non-constant coefficients; solutions involve Legendre functions
- Asymptotic behaviour as $t \to \infty$:

$$k^2 \sim k_x^2 \sim S^2 k_y^2 t^2$$

- ODE has regular singular point at $t = \infty$:

$$\hat{v}_x \propto t^\sigma \qquad\qquad (\hat{v}_y \propto t^{\sigma+1},\ \hat{v}_z \propto t^{\sigma+1},\ \hat{\psi} \propto t^\sigma)$$

- Indicial equation:

$$(\sigma+2)(\sigma+1)S^2 k_y^2 + \kappa^2 k_z^2 = 0$$

$$\sigma = -\frac{3}{2} \pm \left(\frac{1}{4} - \frac{\kappa^2 k_z^2}{S^2 k_y^2}\right)^{1/2}$$

Incompressible fluid dynamics in shearing sheet

$$\hat{v}_x \propto t^\sigma \qquad (\hat{v}_y \propto t^{\sigma+1},\ \hat{v}_z \propto t^{\sigma+1},\ \hat{\psi} \propto t^\sigma)$$

$$\sigma = -\frac{3}{2} \pm \left(\frac{1}{4} - \frac{\kappa^2 k_z^2}{S^2 k_y^2} \right)^{1/2}$$

- Three cases to consider:
  - $\kappa^2 > (k_y^2/k_z^2)(S^2/4)$: $\sigma = -\frac{3}{2} + \mathrm{imaginary}$: $|\hat{\boldsymbol{v}}|^2 \propto t^{-1} \to 0$
  - $0 < \kappa^2 < (k_y^2/k_z^2)(S^2/4)$: $\sigma < -1$: $|\hat{\boldsymbol{v}}|^2 \propto t^{2(\sigma+1)} \to 0$
  - $\kappa^2 < 0$: one root has $\sigma > -1$: $|\hat{\boldsymbol{v}}|^2 \propto t^{2(\sigma+1)} \to \infty$
- Therefore inviscid solutions decay when $\kappa^2 > 0$ but grow (in energy norm) when $\kappa^2 < 0$
- When $\nu > 0$, viscous decay factor E_ν kills off any algebraic growth

Incompressible fluid dynamics in shearing sheet

- Special case of non-rotating shear flow (plane Couette flow)

$$\frac{\mathrm{d}\hat{v}_x}{\mathrm{d}t} = -\mathrm{i}k_x\hat{\psi}$$

$$\frac{\mathrm{d}\hat{v}_y}{\mathrm{d}t} - S\hat{v}_x = -\mathrm{i}k_y\hat{\psi}$$

$$\frac{\mathrm{d}\hat{v}_z}{\mathrm{d}t} = -\mathrm{i}k_z\hat{\psi}$$

$$\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{v}} = 0$$

- Eliminate variables in favour of $\hat{v}_x$ to obtain (after algebra)

$$\frac{\mathrm{d}}{\mathrm{d}t}(k^2\hat{v}_x) = 0$$

Incompressible fluid dynamics in shearing sheet

- Generic non-axisymmetric disturbances $(k_y \neq 0)$:

$$\hat{v}_x \propto k^{-2}\,, \qquad \hat{\psi} \propto k^{-4}$$

  - As $t \to \infty$:

$$\hat{v}_x \propto t^{-2} \qquad \hat{v}_y \to \mathrm{cst}, \qquad \hat{v}_z \to \mathrm{cst}$$

- Generic axisymmetric disturbances $(k_y = 0)$:

$$\hat{\psi} = 0 \qquad \hat{v}_x = \mathrm{cst}\,, \qquad \hat{v}_z = \mathrm{cst}\,, \qquad \mathrm{d}\hat{v}_y/\mathrm{d}t = S\hat{v}_x$$

  - Algebraic growth tempered by viscous decay
  - Kinetic energy grows by a factor $O(\mathrm{Re})^2$ in a time $O(\mathrm{Re})$
  - Reynolds number $\mathrm{Re} = S/\nu k^2$
  - This mechanism plays an essential role in the transition to turbulence in non-rotating shear flows but is suppressed in rotating shear flows because of inertial oscillations

Incompressible fluid dynamics in shearing sheet

- Summary:
  - rotating shear flow is linearly stable when $\kappa^2 > 0$
  - rotating shear flow is linearly unstable when $\kappa^2 < 0$
- Agrees with stability of circular test-particle orbits
- Agrees with Rayleigh's criterion for the linear stability of a cylindrical shear flow $\boldsymbol{u} = r\Omega(r)\,\boldsymbol{e}_\phi$ to axisymmetric perturbations
- The case $\kappa^2 = 0$ (either non-rotating shear flow or one with uniform specific angular momentum) is marginally Rayleigh-stable and allows algebraic growth in the absence of viscosity
- [Discussion of laboratory experiments and numerical simulations]

Incompressible fluid dynamics in shearing sheet

- 2D incompressible dynamics

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)v_x - 2\Omega v_y = -\frac{\partial\psi}{\partial x} + \nu\nabla^2 v_x$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)v_y + (2\Omega - S)v_x = -\frac{\partial\psi}{\partial y} + \nu\nabla^2 v_y$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

- Introduce streamfunction $\chi(x,y,t)$: $v_x = \frac{\partial\chi}{\partial y}, \quad v_y = -\frac{\partial\chi}{\partial x}$
- Instantaneous streamlines are curves $\chi = \mathrm{cst}$
- Vorticity perturbation

$$\boldsymbol{\nabla}\times\boldsymbol{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_z = (-\nabla^2\chi)\boldsymbol{e}_z = \zeta\,\boldsymbol{e}_z$$

Incompressible fluid dynamics in shearing sheet

- Curl of equation of motion (to eliminate pressure):

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\zeta - \cancel{S\frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_x}{\partial x}\frac{\partial v_y}{\partial x}} + \cancel{\frac{\partial v_y}{\partial x}\frac{\partial v_y}{\partial y}}$$

$$\cancel{+(2\Omega - S)\frac{\partial v_x}{\partial x}} - \cancel{\frac{\partial v_x}{\partial y}\frac{\partial v_x}{\partial x}} - \cancel{\frac{\partial v_y}{\partial y}\frac{\partial v_x}{\partial y}} + \cancel{2\Omega\frac{\partial v_y}{\partial y}} = \nu\nabla^2\zeta$$

- Can also be written using Jacobian:

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\zeta = \frac{\partial(\chi,\zeta)}{\partial(x,y)} + \nu\nabla^2\zeta$$

- Solve in conjunction with Poisson equation $\nabla^2\chi = -\zeta$
- Total absolute vorticity is $(2\Omega - S + \zeta)\,\boldsymbol{e}_z$
- Coriolis force drops out of 2D incompressible dynamics!
- Too constrained to allow epicyclic motion / inertial oscillations
- Pure vortex dynamics with background shear

Incompressible fluid dynamics in shearing sheet

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\zeta = \nu\nabla^2\zeta$$

- Multiply by ζ to obtain enstrophy equation

$$\begin{aligned}\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\left(\frac{1}{2}\zeta^2\right) &= \nu\zeta\nabla^2\zeta \\ &= \boldsymbol{\nabla}\cdot(\nu\zeta\boldsymbol{\nabla}\zeta) - \nu|\boldsymbol{\nabla}\zeta|^2\end{aligned}$$

- With suitable boundary conditions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}\zeta^2\,\mathrm{d}A = -\int\nu|\boldsymbol{\nabla}\zeta|^2\,\mathrm{d}A$$

so enstrophy decays

- To maintain vorticity perturbations in the presence of viscosity requires baroclinic or 3D effects or other source terms

Incompressible fluid dynamics in shearing sheet

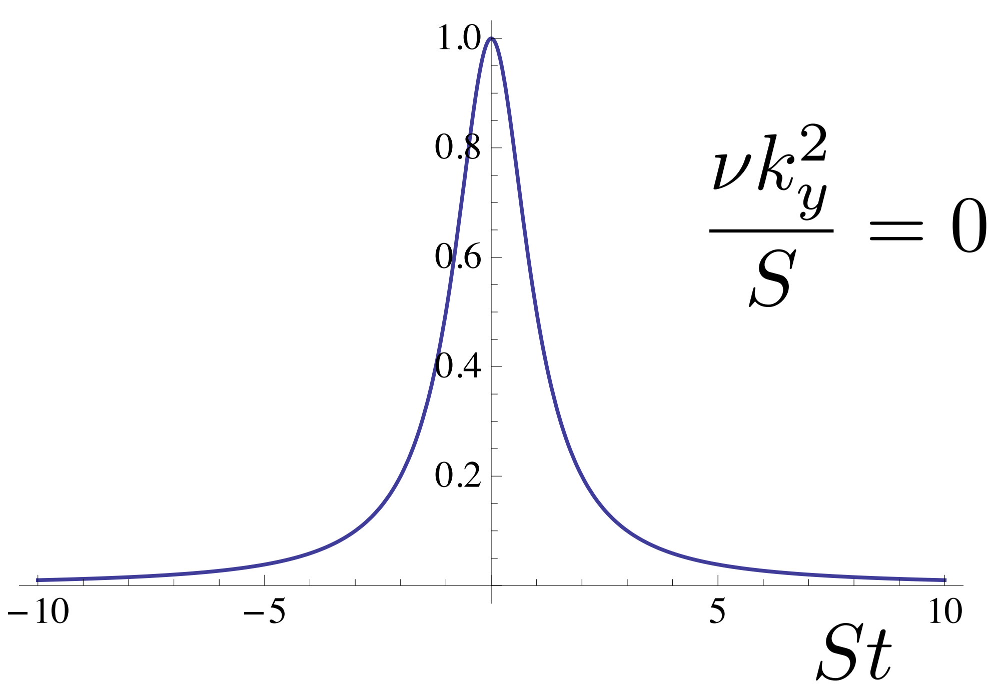
$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\zeta = \nu\nabla^2\zeta$$

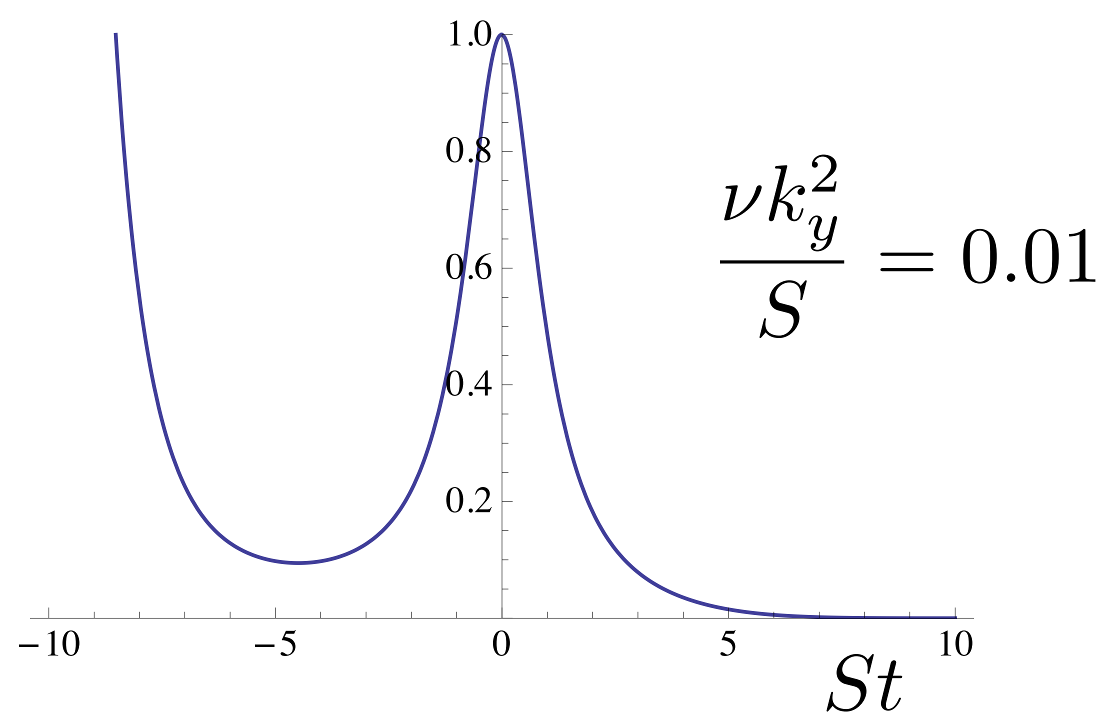
- Shearing-wave solutions $\zeta(\boldsymbol{x},t) = \mathrm{Re}\left\{\tilde{\zeta}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}$:

$$\frac{\mathrm{d}\tilde{\zeta}}{\mathrm{d}t} = -\nu k^2\tilde{\zeta}$$ (nonlinear term vanishes)

$$\tilde{\zeta} \propto E_\nu(t)$$

- Kinetic energy $\propto |\tilde{\boldsymbol{v}}|^2 \propto k^{-2}|\tilde{\zeta}|^2 \propto k^{-2}E_\nu^2$

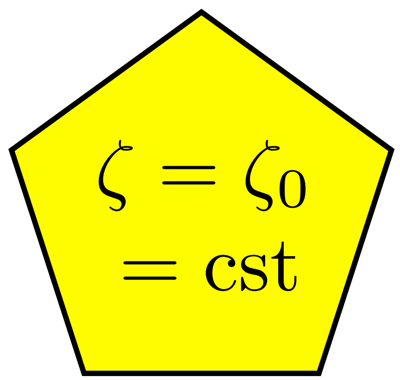




Incompressible fluid dynamics in shearing sheet

- Elliptical vortex patches
- Set $\nu = 0$. Can vorticity resist shear (nonlinear effect)?
- Vortex patch: contour dynamics:

$\zeta = 0$



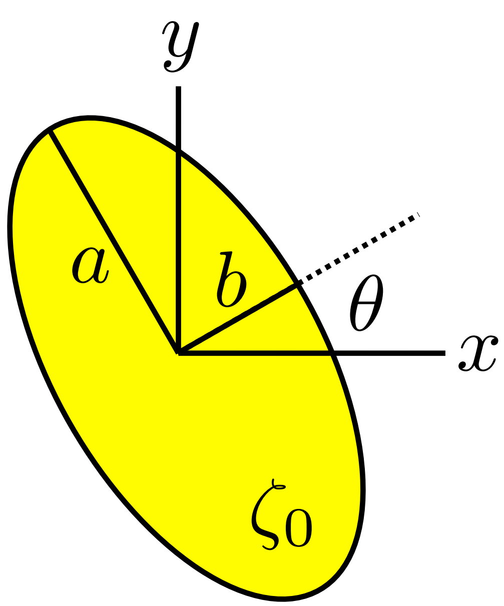
$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0$$

$\zeta \;\rightarrow\; \boldsymbol{v} \;\rightarrow\;$ advection of contour (by $\boldsymbol{u} = \boldsymbol{v} - Sx\,\boldsymbol{e}_y$)

- Do steady solutions exist?

Incompressible fluid dynamics in shearing sheet

- Elliptical vortex patch



- Kirchhoff: $\boldsymbol{v}$ induced by ζ_0 causes ellipse to rotate

  with angular velocity $\dot{\theta} = \dfrac{ab\zeta_0}{(a+b)^2}$

- Shear $\boldsymbol{u}_0 = -Sx\,\boldsymbol{e}_y$ deforms the ellipse according to

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta \qquad \dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2}$$

Incompressible fluid dynamics in shearing sheet

- Elliptical coordinates (ξ,η) : $\xi > 0, \quad 0 \le \eta < 2\pi$

$$\boldsymbol{x} = (x, y) = c(\cosh\xi\cos\eta, \sinh\xi\sin\eta)$$

$$\boldsymbol{h}_\xi = \frac{\partial\boldsymbol{x}}{\partial\xi} = c(\sinh\xi\cos\eta, \cosh\xi\sin\eta)$$

$$\boldsymbol{h}_\eta = \frac{\partial\boldsymbol{x}}{\partial\eta} = c(-\cosh\xi\sin\eta, \sinh\xi\cos\eta) = \boldsymbol{e}_z \times \boldsymbol{h}_\xi$$

- Orthogonal because $\boldsymbol{h}_\xi \cdot \boldsymbol{h}_\eta = 0$
- Scale factors given by

$$\begin{aligned} h_\xi^2 = |\boldsymbol{h}_\xi|^2 &= c^2(\sinh^2\xi\cos^2\eta + \cosh^2\xi\sin^2\eta) \\ &= c^2[\sinh^2\xi(1-\sin^2\eta) + (1+\sinh^2\xi)\sin^2\eta] \\ &= c^2(\sinh^2\xi + \sin^2\eta) \end{aligned}$$

$$h_\eta^2 = |\boldsymbol{h}_\eta|^2 = |\boldsymbol{h}_\xi|^2 = h_\xi^2$$

- Coordinate singularities $\xi = 0, \ \eta = 0 \text{ or } \pi$

Incompressible fluid dynamics in shearing sheet

- Elliptical coordinates (ξ,η) : $\xi > 0, \quad 0 \le \eta < 2\pi$

$$\boldsymbol{x} = (x, y) = c(\cosh\xi\cos\eta, \sinh\xi\sin\eta)$$

- Curves of constant ξ :

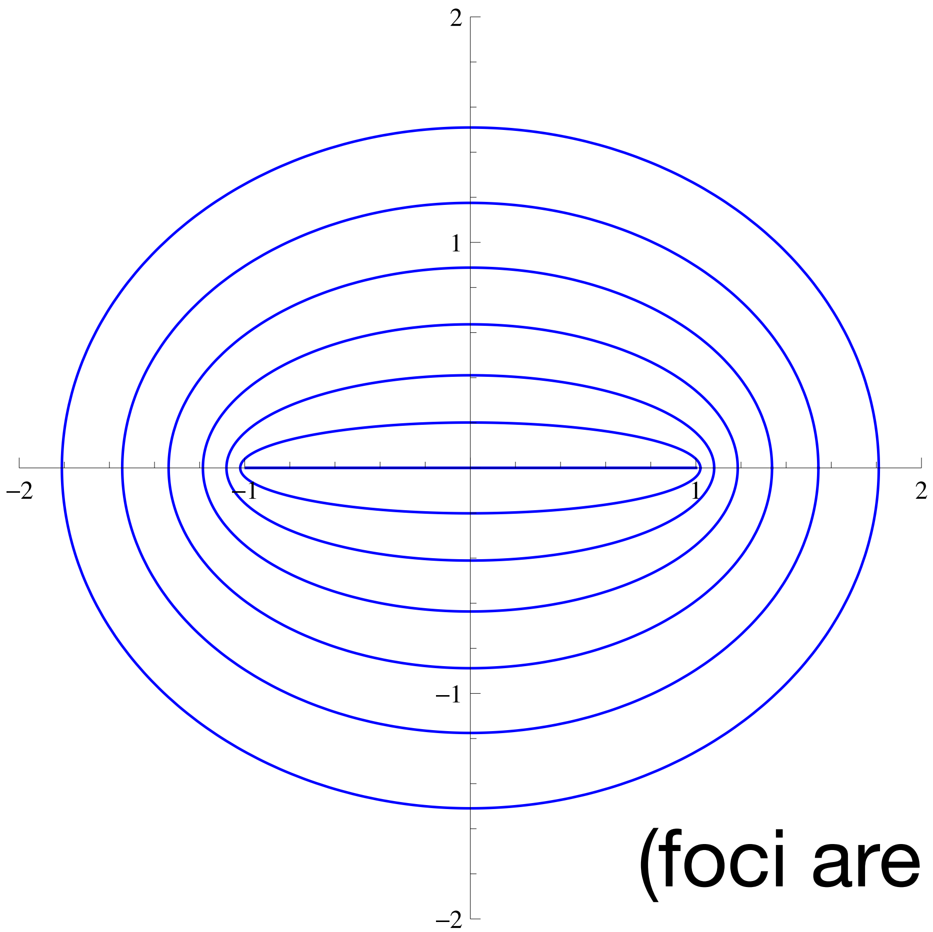
$$\left(\frac{x}{c\cosh\xi}\right)^2 + \left(\frac{y}{c\sinh\xi}\right)^2 = 1$$

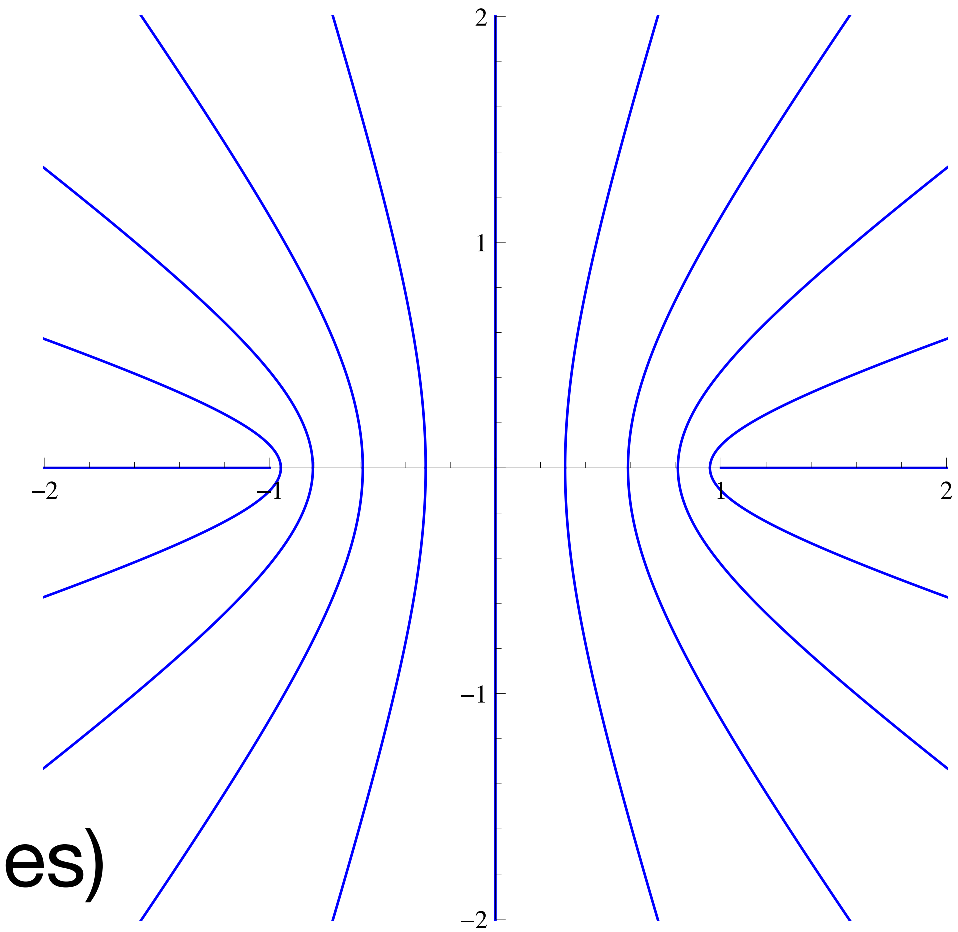
confocal ellipses

- Curves of constant η :

$$\left(\frac{x}{c\cos\eta}\right)^2 - \left(\frac{y}{c\sin\eta}\right)^2 = 1$$

confocal hyperbolae

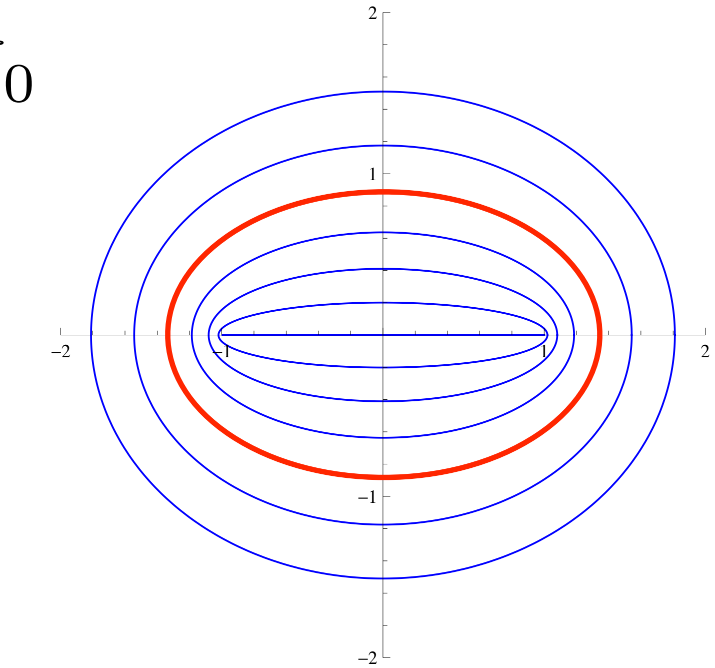




(foci are singularities)

Incompressible fluid dynamics in shearing sheet

- Let the boundary of the vortex patch at some instant be the ellipse $\xi = \xi_0$ with semi-axes $a = c\cosh\xi_0\,,\ b = c\sinh\xi_0$



- Laplacian of streamfunction

$$\nabla^2\chi = \frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta}{h_\xi}\frac{\partial\chi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\xi}{h_\eta}\frac{\partial\chi}{\partial\eta}\right)\right]$$

$$= \frac{1}{h_\xi^2}\left(\frac{\partial^2\chi}{\partial\xi^2} + \frac{\partial^2\chi}{\partial\eta^2}\right)$$

(conformal transformation)

- Poisson's equation $\nabla^2\chi = -\zeta$ becomes:

$$\frac{\partial^2\chi}{\partial\xi^2} + \frac{\partial^2\chi}{\partial\eta^2} = \begin{cases} -\zeta_0 c^2(\sinh^2\xi + \sin^2\eta), & 0 < \xi < \xi_0 \\ 0, & \xi > \ 0 \end{cases}$$

- Fourier series: $\sinh^2\xi + \sin^2\eta = \frac{1}{2}(\cosh 2\xi - \cos 2\eta)$

Incompressible fluid dynamics in shearing sheet

- Particular integral for $0 < \xi < \xi_0$ is $-\frac{1}{8}\zeta_0 c^2(\cosh 2\xi + \cos 2\eta)$
- Complementary functions:

$$\{\cos(n\eta) \text{ or } \sin(n\eta)\} \times \{\cosh(n\xi) \text{ or } \sinh(n\xi)\} \qquad n = 1, 2, 3, \dots$$

$$\{1 \text{ or } \eta\} \times \{1 \text{ or } \xi\} \qquad n = 0$$

- Only $n = 0, 2$ present in solution, which is even in η
- For $\xi > \xi_0$, require decaying solution $\exp(-2\xi)$ in the case $n = 2$
- For $\xi < \xi_0$, solutions ξ and $\sinh 2\xi \cos 2\eta$ are singular at foci
- Solution is therefore of the form

$$\chi = \begin{cases} -\frac{1}{8}\zeta_0 c^2(\cosh 2\xi + \cos 2\eta) + C_1 + C_2 \cosh 2\xi \cos 2\eta, & 0 < \xi < \xi_0 \\ C_3 + C_4\xi + C_5 \exp(-2\xi) \cos 2\eta, & \xi > \xi_0 \end{cases}$$

- Find constants by requiring continuity of χ and $\partial\chi/\partial\xi$ (velocity is continuous)

Incompressible fluid dynamics in shearing sheet

$$\frac{\chi}{-\frac{1}{8}\zeta_0 c^2} = \begin{cases} \cosh 2\xi + \cos 2\eta - \exp(-2\xi_0)\cosh 2\xi \cos 2\eta, & 0 < \xi < \xi_0 \\ \cosh 2\xi_0 + \sinh 2\xi_0 [2(\xi - \xi_0) + \exp(-2\xi)\cos 2\eta], & \xi > {}_0\xi \end{cases}$$

- To find the normal velocity on the boundary, evaluate χ at $\xi = \xi_0$:

$$\chi_0(\eta) = \mathrm{cst} - \tfrac{1}{8}\zeta_0 c^2 \sinh 2\xi_0 \exp(-2\xi_0)\cos 2\eta$$

- Compare with the streamfunction of a uniformly rotating flow:

$$\chi_\omega = -\tfrac{1}{2}\omega(x^2 + y^2)$$

$$\chi_\omega(\eta) = -\tfrac{1}{2}\omega c^2(\cosh^2\xi_0 \cos^2\eta + \sinh^2\xi_0 \sin^2\eta)$$

$$= \mathrm{cst} - \tfrac{1}{4}\omega c^2 \cos 2\eta$$

- Identify Kirchhoff rotation of vortex:

$$\omega = \tfrac{1}{2}\zeta_0 \sinh 2\xi_0 \exp(-2\xi_0)$$

$$= \frac{ab\zeta_0}{(a+b)^2}$$

using $a + b = c\cosh\xi_0 + c\sinh\xi_0 = c\exp\xi_0$

Incompressible fluid dynamics in shearing sheet

- Alternative: streamfunction inside vortex:

$$-\tfrac{1}{8}\zeta_0 c^2[\cosh 2\xi + \cos 2\eta - \exp(-2\xi_0)\cosh 2\xi \cos 2\eta]$$

$$= -\tfrac{1}{8}\zeta_0 \left[2(x^2+y^2) - \left(\frac{a-b}{c}\right)^2 (2x^2 - 2y^2 - c^2)\right]$$

$$= \mathrm{cst} - \tfrac{1}{2}\zeta_0 \left(\frac{ab}{a+b}\right)^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \tfrac{1}{2}\frac{ab\zeta_0}{(a+b)^2}(x^2+y^2)$$

concentric elliptical flow uniform rotation

$$\omega = \frac{ab\zeta_0}{(a+b)^2}$$

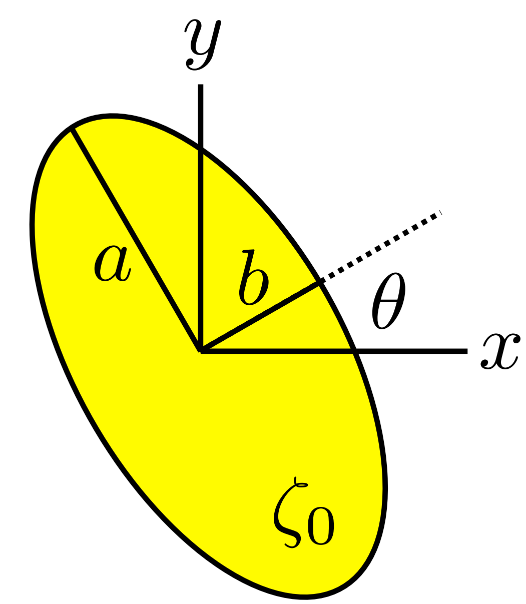
Incompressible fluid dynamics in shearing sheet

- Uniform shear flow $\dot{x} = 0\,,\ \dot{y} = -Sx$ induces a linear mapping

$$x = x_0\,,\ y = y_0 - Sx_0 t$$

- Ellipse centred on the origin remains so (quadratic curve)

$$\left(\frac{-x\sin\theta + y\cos\theta}{a}\right)^2 + \left(\frac{x\cos\theta + y\sin\theta}{b}\right)^2 = 1$$



- Differentiate with respect to t
(in Lagrangian sense) to find (after algebra):

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta$$

$$\dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2}$$

Incompressible fluid dynamics in shearing sheet

- Combine effects:

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta$$

$$\dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2} + \frac{ab\zeta_0}{(a+b)^2}$$

- Area πab is conserved. Rewrite in terms of aspect ratio $r = \dfrac{a}{b}$:

$$\frac{\dot{r}}{r} = 2S\sin\theta\cos\theta$$

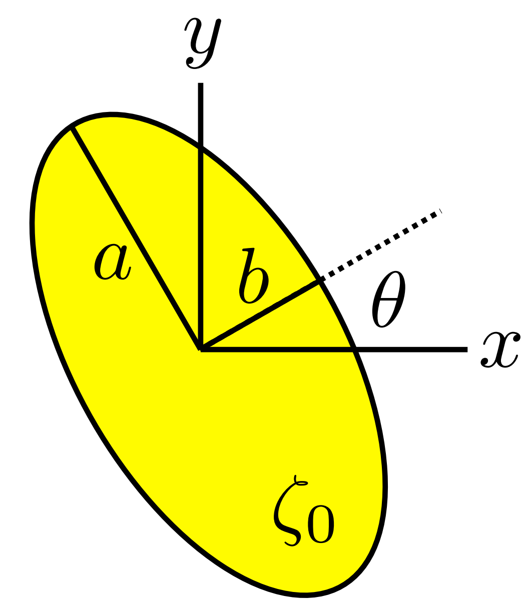
$$\dot{\theta} = \frac{S(\cos^2\theta - r^2\sin^2\theta)}{r^2 - 1} + \frac{r\,\zeta_0}{(r+1)^2}$$

- 2D autonomous dynamical system
- Chaplygin (1899); Moore & Saffman (1971); Kida (1981)
- Note that ζ_0 is the vorticity perturbation relative to the background

Incompressible fluid dynamics in shearing sheet

$$\frac{\dot r}{r} = 2S\sin\theta\cos\theta$$

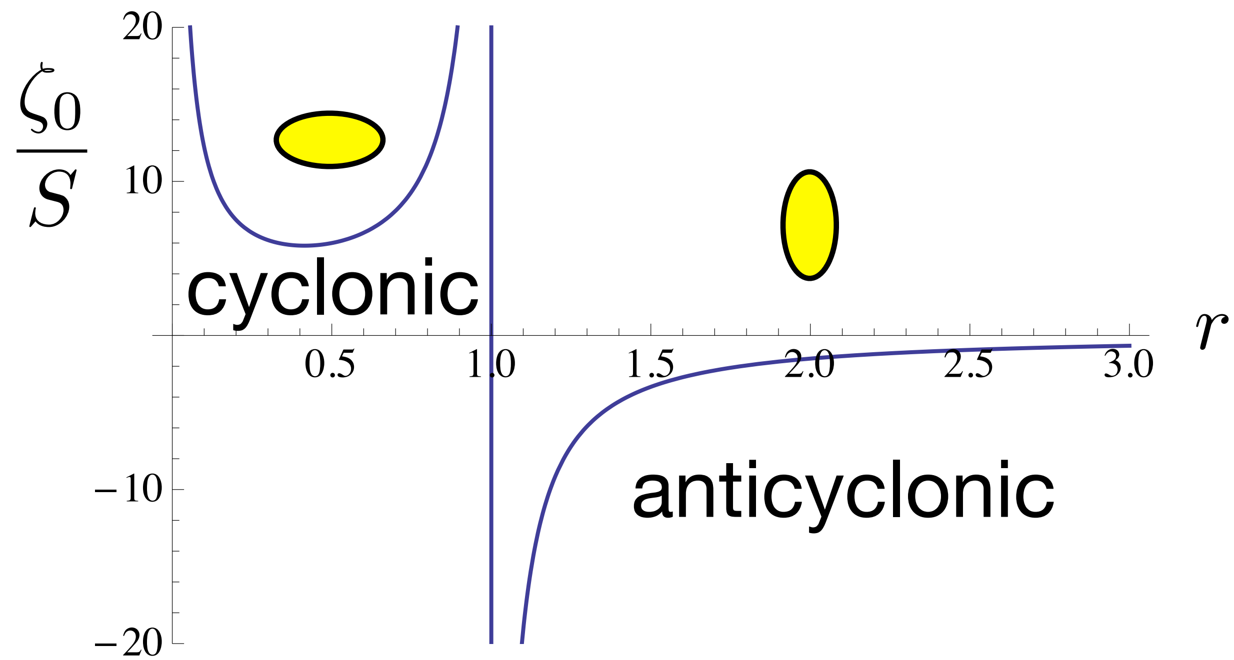
$$\dot\theta = \frac{S(\cos^2\theta - r^2\sin^2\theta)}{r^2-1} + \frac{r\,\zeta_0}{(r+1)^2}$$



- Fixed points:

$\theta = 0$ without loss of generality (let $r<1$ if need be)

$$\frac{S}{r^2-1} + \frac{r\,\zeta_0}{(r+1)^2} = 0 \quad\Rightarrow\quad \frac{\zeta_0}{S} = -\frac{(r+1)}{r(r-1)}$$



Incompressible fluid dynamics in shearing sheet

- Stability of fixed point $\theta = 0$: linearized equations:

$$\dot{\delta r} = 2Sr\,\delta\theta$$

$$\dot{\delta\theta} = S\,\delta\,r\frac{\partial}{\partial r}\left[\frac{1}{r^2-1} + \frac{r}{(r+1)^2}\frac{\zeta_0}{S}\right] = S\,\delta\,r\frac{\partial f}{\partial r}$$

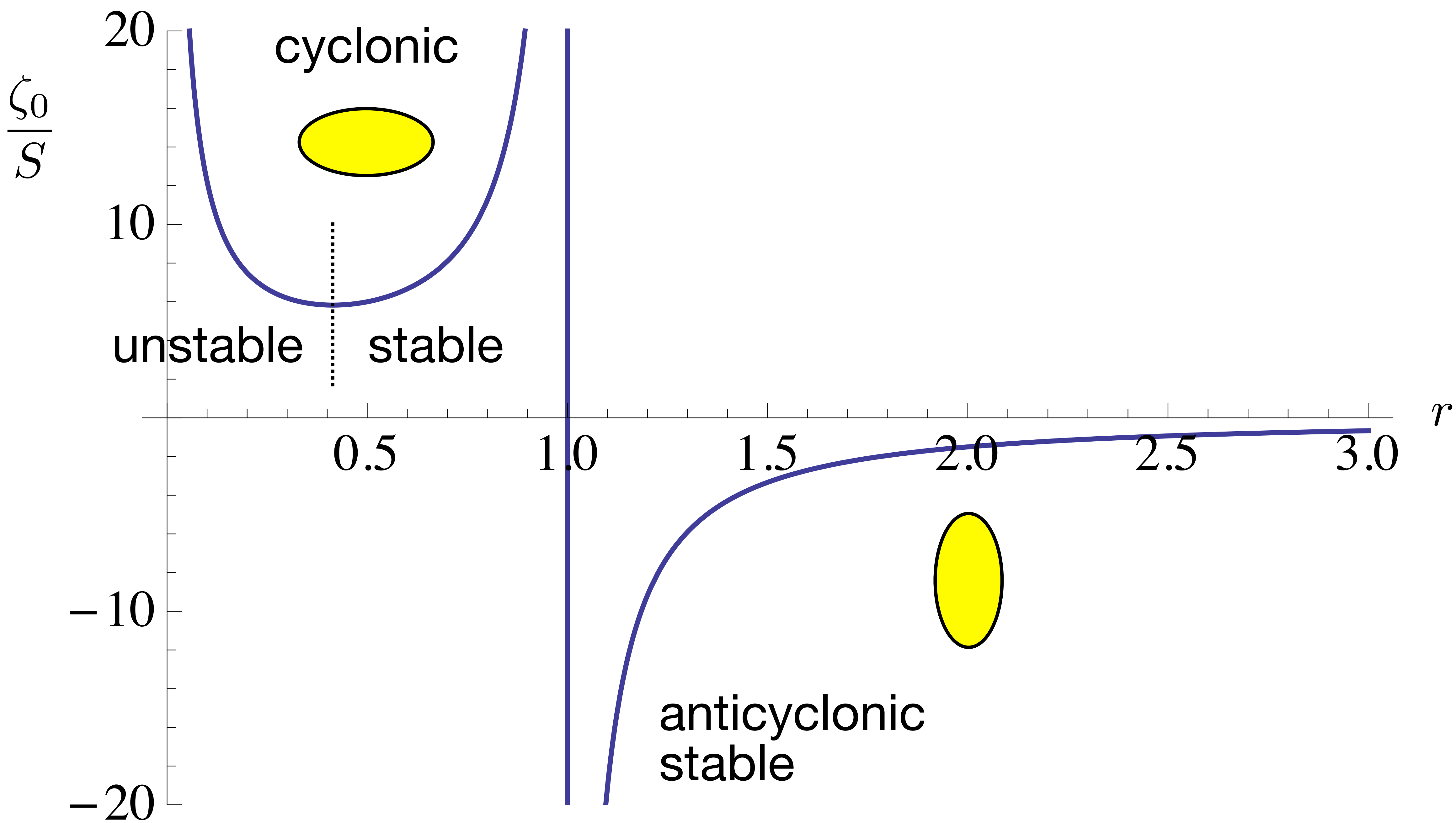
($f = 0$ at equilibrium)

$$\Rightarrow \quad \ddot{\delta r} = 2S^2 r\frac{\partial f}{\partial r}\,\delta r \qquad\qquad \frac{\partial f}{\partial r} = -\frac{(r^2+2r-1)}{r(r^2-1)^2}$$

- Unstable if $\dfrac{\partial f}{\partial r} > 0$, i.e. $r < \sqrt{2}-1$
- Stable if $\dfrac{\partial f}{\partial r} < 0$, i.e. $r > \sqrt{2}-1$

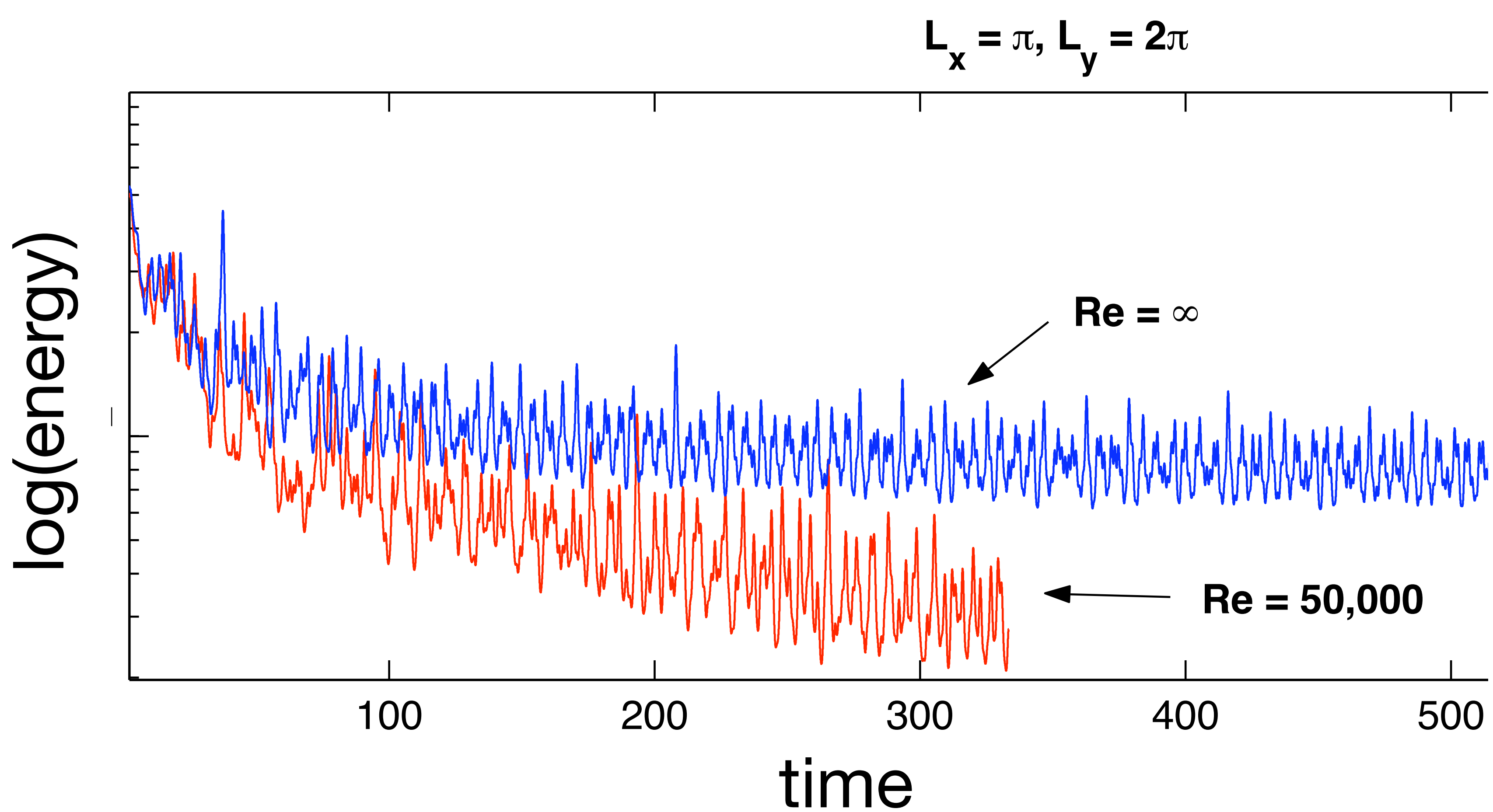
- Other instabilities exist, e.g. elliptical instability (3D)

Incompressible fluid dynamics in shearing sheet

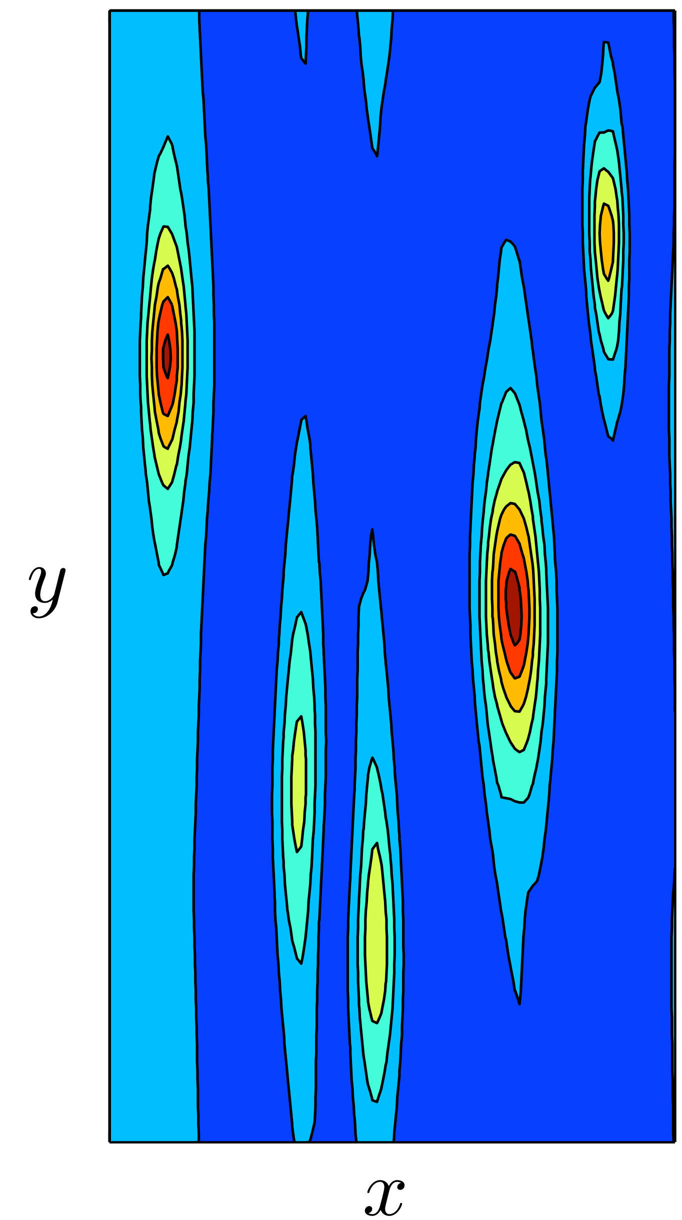


Incompressible fluid dynamics in shearing sheet

- Nonlinear simulations (Umurhan & Regev 2004)



Umurhan, O. M. and Regev, O., 2004, A & A, 427, 3, 864



vorticity snapshot

Incompressible fluid dynamics in shearing sheet

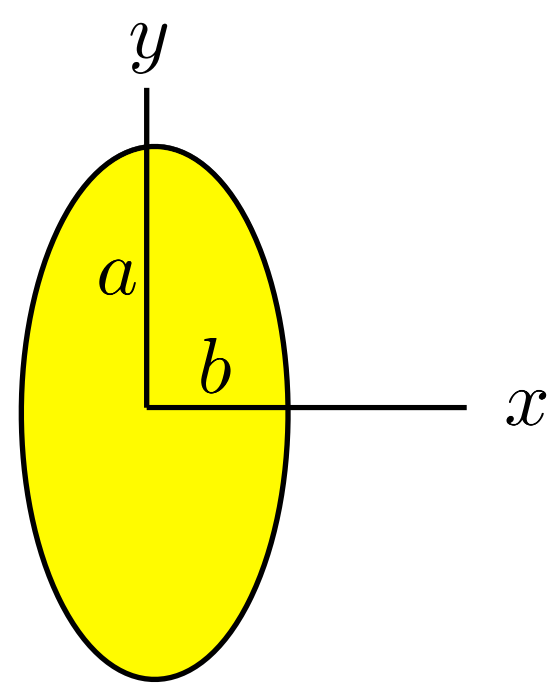
- Particle dynamics in core of steady elliptical vortex

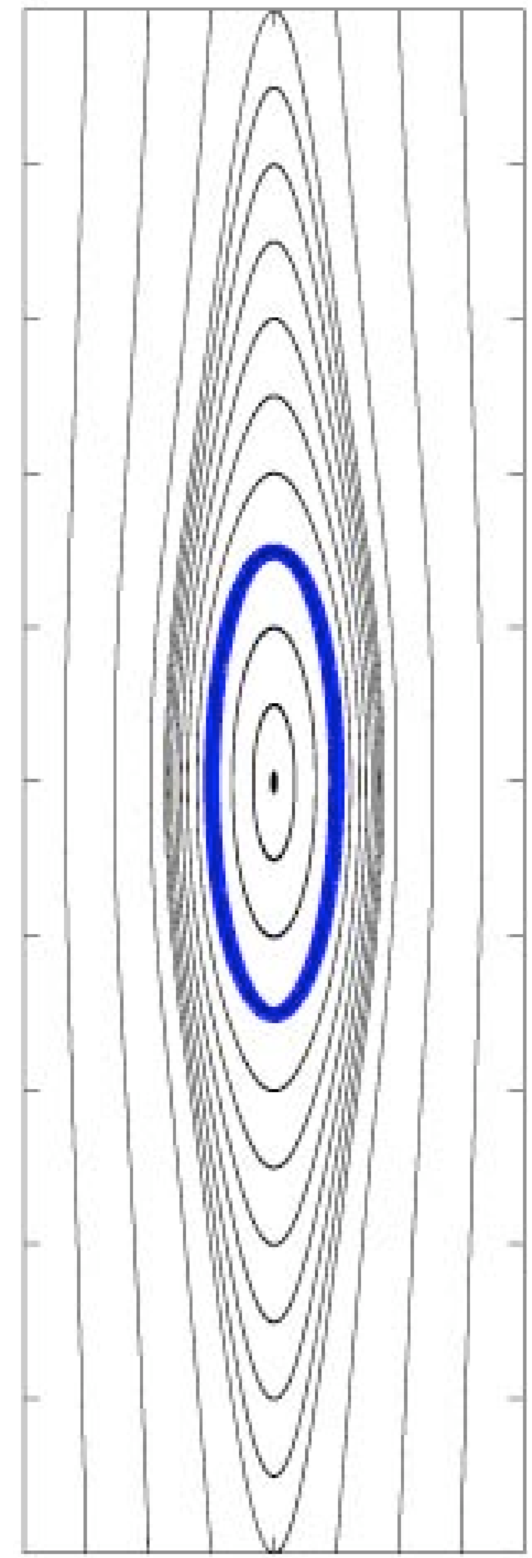
- Total streamfunction
  (nested elliptical streamlines)

$$\propto \frac{x^2}{b^2} + \frac{y^2}{a^2}$$

$$\propto r^2 x^2 + y^2$$

so $\boldsymbol{u} \propto (y, -r^2 x)$





Lesur, G. and Papaloizou, J. C. B., 2009, A & A, 498, 1, 3

- Let $\boldsymbol{u} = A\left(\frac{y}{r}, -rx\right)$

$$(\boldsymbol{\nabla} \times \boldsymbol{u})_z = -A\left(\frac{1}{r} + r\right) = -S + \zeta_0 = -\frac{(r^2+1)}{r(r-1)} S$$

$$\Rightarrow \quad A = \frac{S}{r-1}$$

Incompressible fluid dynamics in shearing sheet

- Motion of particle subject to drag force:

$$\ddot{x} - 2\Omega\dot{y} = 2\Omega S x - \gamma(\dot{x} - u_x)$$

$$\ddot{y} + 2\Omega\dot{x} = -\gamma(\dot{y} - u_y)$$

$$\boldsymbol{u} = A\left(\frac{y}{r}, -rx\right)$$

$$A = \frac{S}{r-1}$$

- Linear system: solutions $x,\, y \propto \mathrm{e}^{\lambda t}$:

$$(\lambda^2 - 2\Omega S + \gamma\lambda)x = (2\Omega\lambda + \gamma A r^{-1})y$$

$$(\lambda^2 + \gamma\lambda)y = -(2\Omega\lambda + \gamma A r)x$$

$$(\lambda^2 - 2\Omega S + \gamma\lambda)(\lambda^2 + \gamma\lambda) + (2\Omega\lambda + \gamma A r^{-1})(2\Omega\lambda + \gamma A r) = 0$$

$$\lambda^4 + 2\gamma\lambda^3 + (4\Omega^2 - 2\Omega S + \gamma^2)\lambda^2 + \left[-2\Omega S + 2\Omega A(r + r^{-1})\right]\gamma\lambda + \gamma^2 A^2 = 0$$

$$\lambda^4 + 2\gamma\lambda^3 + (\kappa^2 + \gamma^2)\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0$$

Incompressible fluid dynamics in shearing sheet

$$\lambda^4 + 2\gamma\lambda^3 + (\kappa^2 + \gamma^2)\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0$$

- Limit of small γ (weak drag; large particles):
  - $\lambda \sim \pm\mathrm{i}\kappa + c_1\gamma + O(\gamma^2)$ $\qquad c_1 = -1 - \dfrac{\Omega\zeta_0}{\kappa^2}$
  - $\lambda \sim c_2\gamma + O(\gamma^2)$ $\qquad c_2 = \dfrac{\Omega\zeta_0}{\kappa^2} \pm \left(\dfrac{\Omega^2\zeta_0^2}{\kappa^4} - \dfrac{A^2}{\kappa^2}\right)^{1/2}$
  - For stability (decay to centre), require

    $-\kappa^2 < \Omega\zeta_0 < 0$ (must be anticyclonic)
- Limit of large γ (strong drag; small particles):
  - $\lambda \sim c_3\gamma + O(1)$ $\qquad c_3 = -1$
  - $\lambda \sim \pm\mathrm{i}A + c_4\gamma^{-1} + O(\gamma^{-2})$ $\qquad c_4 = \Omega\zeta_0 + A^2$
  - For stability (decay to centre), require

    $\Omega\zeta_0 < -A^2$

Incompressible fluid dynamics in shearing sheet

$$\lambda^4 + 2\gamma\lambda^3 + (\kappa^2 + \gamma^2)\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0$$

- For general γ, when does marginal stability occur?
  - $\lambda = 0$: never
  - $\lambda = -\mathrm{i}\omega, \omega \in \mathbf{R},\ \omega \neq 0$:

$$\omega^4 - (\kappa^2 + \gamma^2)\omega^2 + \gamma^2 A^2 = 0$$

$$2\gamma\omega^3 + 2\Omega\zeta_0\gamma\omega = 0$$

$$\Rightarrow \omega^2 = -\Omega\zeta_0 \ (> 0)$$ (must be anticyclonic)

$$(\Omega\zeta_0)^2 + (\kappa^2 + \gamma^2)\Omega\zeta_0 + \gamma^2 A^2 = 0$$

  - LHS is negative for all γ , so all particles decay to centre, if

$$A^2 < -\Omega\zeta_0 < \kappa^2$$ (agrees with two limits considered)

$$\frac{S^2}{(r-1)^2} < \frac{(r+1)\Omega S}{r(r-1)} < 2\Omega(2\Omega - S)$$

Incompressible fluid dynamics in shearing sheet

$$\frac{S^2}{(r-1)^2} < \frac{(r+1)\Omega S}{r(r-1)} < 2\Omega(2\Omega - S)$$

- Keplerian disc:

$$\frac{9}{4}\frac{1}{(r-1)^2} < \frac{3}{2}\frac{(r+1)}{r(r-1)} < 1$$

$$\Rightarrow \quad r > 3$$

- Alternative method: Routh–Hurwitz criterion:

  The roots of the real polynomial $\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d$
  all have $\mathrm{Re}(\lambda) < 0$ if and only if $a, b, c, d > 0 \quad \text{and} \quad abc > a^2 d + c^2$

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