Fast and Slow: The Dynamics of Superrotation Phenomena in Planetary Atmospheres: Introduction

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Aims & Motivating Questions

- Meaning and concept of super-rotation: How should it be defined?
- Where does it occur, and when?
  - Explore almost every known planet, not just Earth!
- What dynamical processes and mechanisms can generate and maintain it?
- Is there a general scaling theory for atmospheric super-rotation?
- Which particular processes and mechanisms are responsible for the observed super-rotation on Earth, other terrestrial planets and the gas and ice giants?
  - Implications for planetary climate?
- What outstanding issues remain for further research?

Overall plan

I. Fundamental concepts
   - Definitions, constraints, observations (1st look), analysis of an analogue system

II. Simple models and theories
   - Simple conceptual models, scaling arguments, wave-mean flow interactions

III. Super-rotation mechanisms in simple systems
   - Laboratory experiments, simplified numerical models

IV. Super-rotation in fast-rotating planets
   - Earth, Mars, Gas and Ice Giants

V. Super-rotation in slowly-rotating planets
   - Titan, Venus, tidally-locked exoplanets
   - Synthesis and discussion…..

Fast and Slow: The Dynamics of Superrotation Phenomena in Planetary Atmospheres: I. Fundamental concepts

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Plan

- What is super-rotation?
- Constraints on axisymmetric circulation: Hide’s theorem(s)
- Observations of super-rotation around the Solar System
- Super-rotation in an analogous system: axisymmetric flow in a rotating, cylindrical annulus
- A scaling theory for annulus flows

What is super-rotation?

- Earliest reference that mentions it by H Rishbeth (Nature 1971) on rotation of Earth's upper atmosphere (thermosphere and ionosphere)
- Based on observations in the 1960s of satellite orbits (King-Hele 1964) at low altitude (~300 km)
- Super-rotation defined in terms of ratio of **angular velocities**
- $\Lambda = \dfrac{\left[\Omega + \dfrac{U}{(R+h)\cos\varphi}\right]}{\Omega} = 1 + \dfrac{U}{(R+h)\Omega\cos\varphi}$

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Rotation of the Variation of Upper Atmosphere

FROM observations of satellite orbits it has been deduced that the atmosphere above about 200 km altitude rotates 20-30% faster than the Earth, so that there exists a net west-to-east wind of order 100 m s^{-1} (refs. 1, 2). This "super-rotation" has not yet been satisfactorily explained. The wind systems driven by the diurnal heating and cooling of the thermosphere[3] do not produce any significant net rotation at mid-latitudes[4]. The satellite data are somewhat weighted towards low latitudes so the rotation may be most pronounced there; calculations suggest that the thermospheri… rotation at low latitudes[5], th… account fully for the observati… by dynamo electric fields in th… any significant difference to t…

From Rishbeth 2002

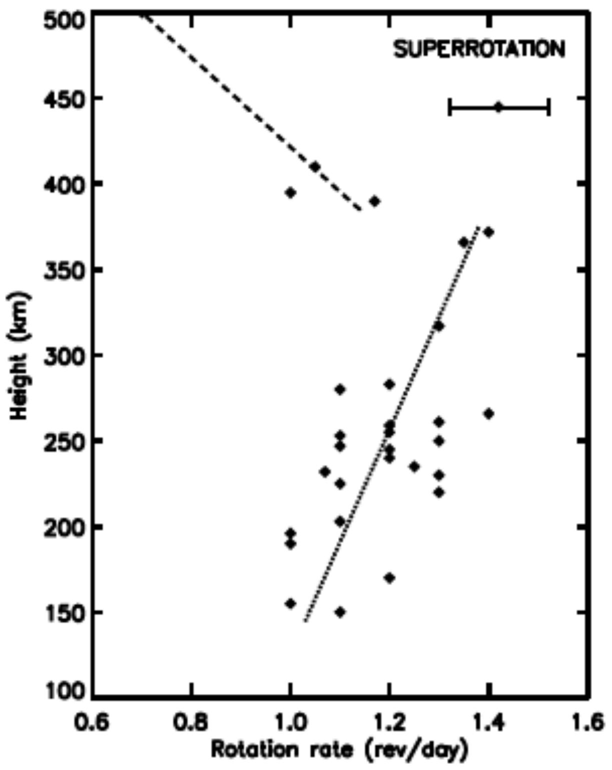


Fig. 1. The superrotation ratio Λ as derived from observations of satellite orbits, based on the data of King-Hele (1971), showing a typical error bar of ±0.1. A few points have larger (±0.15) or smaller error bars (±0.05).

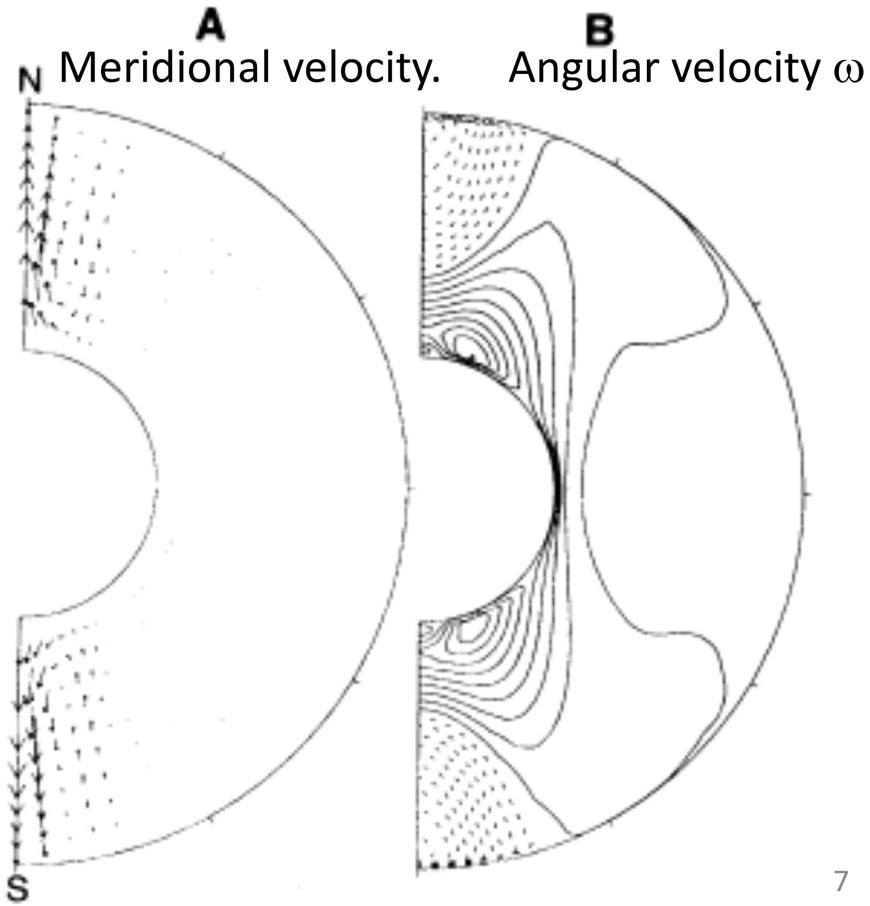
What is super-rotation?

- Earth's inner core is also described as "super-rotating" by Glatzmaier & Roberts (1996 *Science*) from numerical simulations of MHD convection in the outer core
- Interpreted as a positive angular velocity relative to the rest of the Earth
- Maintained mainly via EM torques from motion of outer core….
- But equatorial region of outer core sub-rotates except close to inner core boundary…..

Rotation and Magnetism of Earth's Inner Core

Gary A. Glatzmaier* and Paul H. Roberts

Three-dimensional numerical simulations of the geodynamo suggest that a super-rotation of Earth's solid inner core relative to the mantle is maintained by magnetic coupling between the inner core and an eastward thermal wind in the fluid outer core. This mechanism, which is analogous to a synchronous motor, also plays a fundamental role in the generation of Earth's magnetic field.



What is remarkable about positive angular **velocity**?

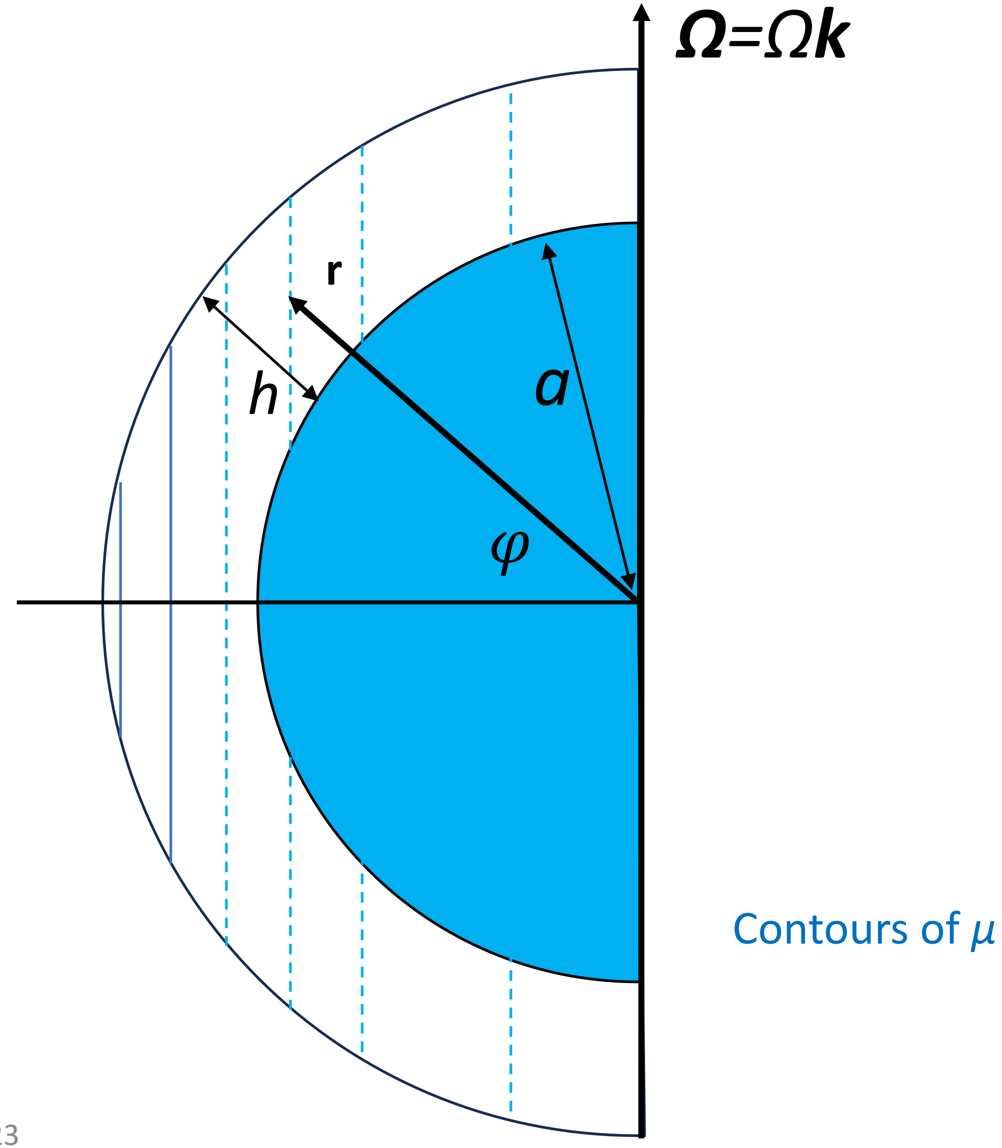
- Doesn't necessarily require special forces to act, e.g. if **angular momentum** is conserved.

$$J = I\omega = const.$$

- Example: Ice skater reduces her moment of inertia and increases her angular velocity ω
- Better to define super-rotation with respect to angular momentum?

Angular momentum of an atmosphere in solid-body co-rotation

- Axial angular momentum defined as
  - $\mu = (\boldsymbol{r} \times \boldsymbol{p_\theta}).\boldsymbol{k} = \rho\Omega[(a+h)\cos\varphi]^2$
- Increases with cylindrical radius $r\cos\varphi$ from rotation axis
- Maximum at the equator
  - $\mu_{max} = \rho\Omega(a+h)^2$
  - $\approx \rho\Omega a^2$ for $h \ll a$
- Define super-rotation as
  - $\mu = \rho[u + \Omega(a+h)\cos\varphi](a+h)\cos\varphi > \mu_{max}$
  - Occurs e.g. when *u > 0* on the equator.



When is angular momentum conserved?

- Consider zonal momentum equation in spherical coordinates for a shallow atmosphere (*h* << *a*)

  - $$\frac{\partial u}{\partial t} + \left[\frac{u}{a\cos\varphi}\frac{\partial u}{\partial \lambda}\right] + \frac{v}{a}\frac{\partial u}{\partial \varphi} + w\frac{\partial u}{\partial z} - \frac{uv\tan\varphi}{a} - 2\Omega v\sin\varphi = -\left[\frac{1}{\rho a}\frac{\partial p}{\partial \lambda}\right] + F_B \qquad (1)$$

    - λ is longitude, F_B represents body forces (friction, viscosity, MHD etc…)

- Define specific axial angular momentum (per unit mass)

  - $$m = \frac{\mu}{\rho} = [u + \Omega a\cos\varphi]a\cos\varphi \qquad (2)$$

- So

  - $$\frac{\partial m}{\partial t} = \frac{\partial u}{\partial t}a\cos\varphi$$
    $$= -\left\{v\cos\varphi\frac{\partial u}{\partial \varphi} + wa\cos\varphi\frac{\partial u}{\partial z} - uv\sin\varphi - 2\Omega av\cos\varphi\sin\varphi - F_B a\cos\varphi\right\} \qquad (3a)$$

    - [neglecting terms in $\partial/\partial\lambda$ i.e. assuming axisymmetry]

    $$= -\left\{\frac{v}{a}\frac{\partial m}{\partial \varphi} + w\frac{\partial m}{\partial z}\right\} + F_B a\cos\varphi \qquad (3b)$$

When is angular momentum conserved?

- Hence for a circulation that is symmetric about the rotation axis
  - $\frac{\partial m}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} m = F_B a \cos\varphi$ (4)
- *m* is materially conserved in the absence of body torques and friction by an axisymmetric circulation.
- Conversely, *m* cannot exceed $m_{max} = \Omega a^2$ in an axisymmetric flow without non-axisymmetric eddies (or body forces)
  - Zonal pressure torques etc.....
- Often referenced as Hide's theorem (J. Atmos. Sci. 1969)

Raymond Hide [1937 – 2016]

How to measure/quantify super-rotation?

- Suggests a way to quantify super-rotation as the degree to which Hide's ***local*** limit on m (or its zonal mean $\overline{m}$) is exceeded (Read QJRMS 1986).
- Define a dimensionless local super-rotation index

$$s = \frac{m - \Omega a^2}{\Omega a^2} = \frac{u \cos \varphi}{\Omega a} \qquad (5)$$

- Implies $s \leq 0$ for axisymmetric circulations but s can be ≥ 0 for non-axisymmetric flows under certain conditions

Aside: compressible or incompressible?

- $\frac{\partial m}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} m = F_B a \cos\varphi$

$$= \frac{\partial m}{\partial t} + \nabla.(\boldsymbol{u}m) \text{ for either} \qquad (6)$$

  - (i) incompressible flow $[\nabla.\boldsymbol{u} = 0]$ or
  - (ii) compressible flow using *p* as vertical coordinate
- For compressible flow with height *z* as vertical coordinate
- $\frac{\partial(\rho m)}{\partial t} + \nabla.(\boldsymbol{u}\rho m) = F_B \rho a \cos\varphi$

$$= \frac{\partial \mu}{\partial t} + \nabla.(\boldsymbol{u}\mu) \qquad (7)$$

How to measure/quantify super-rotation?

- In light of (6) or (7), if *m* is materially conserved everywhere [in a closed system] then the total volume- [or mass-]integrated AM must also be invariant.
- Hence we can also define a dimensionless global super-rotation index (Read 1986)

$$S = \frac{\iiint m - \Omega(a\cos\varphi)^2 dV}{\iiint \Omega(a\cos\varphi)^2 dV} \text{ or } \frac{\iiint \rho[m - \Omega(a\cos\varphi)^2]dV}{M_0} \qquad (8)$$

  - [Or its compressible equivalent based on (7)]
  - Where M_0 *is* the total integrated (mass-weighted) AM for an atmosphere in solid-body co-rotation with the planet at Ω
- *S* = 0 for an atmosphere that does not exchange AM with other parts of the planet and/or experiences no external body torques

Where do we observe super-rotation (and how strong)?

Earth - Annual mean

[zonally averaged] circulation

[As observed - ERA40
- Kallberg et al. 2004]

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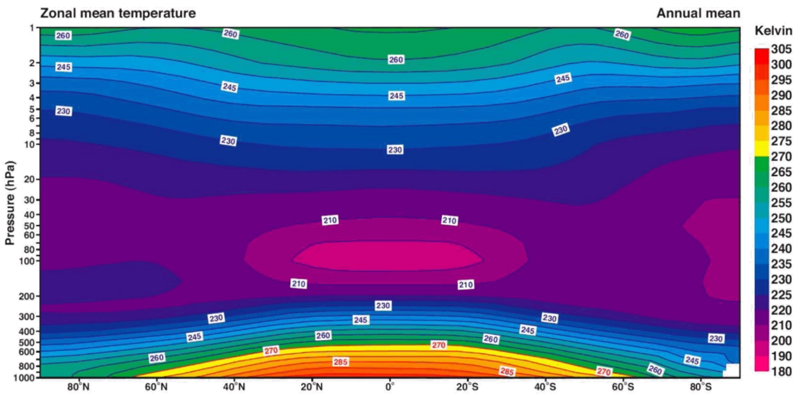
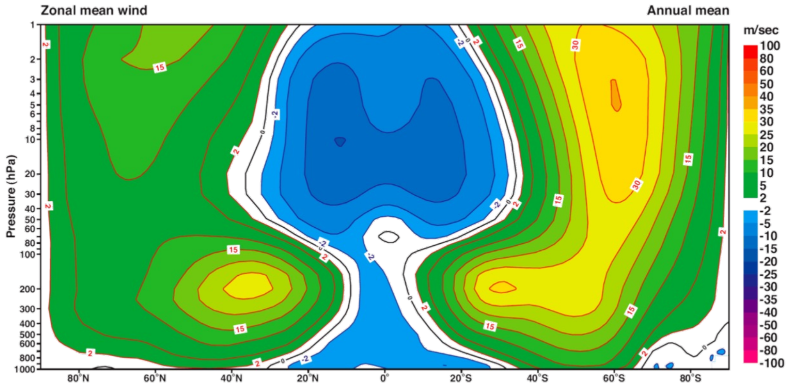


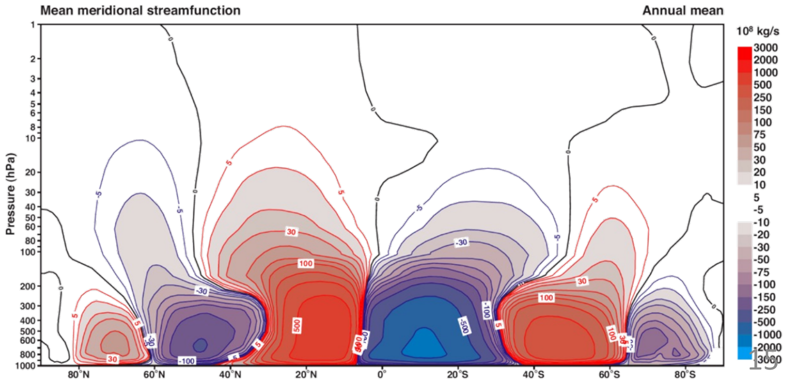
(a)

U



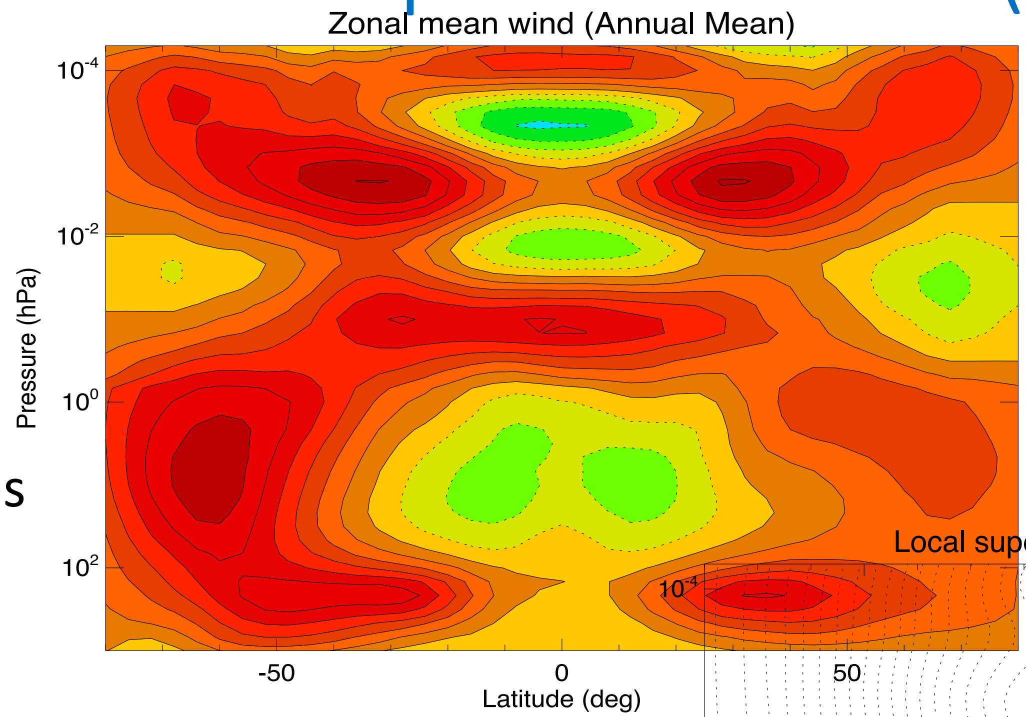
(b)

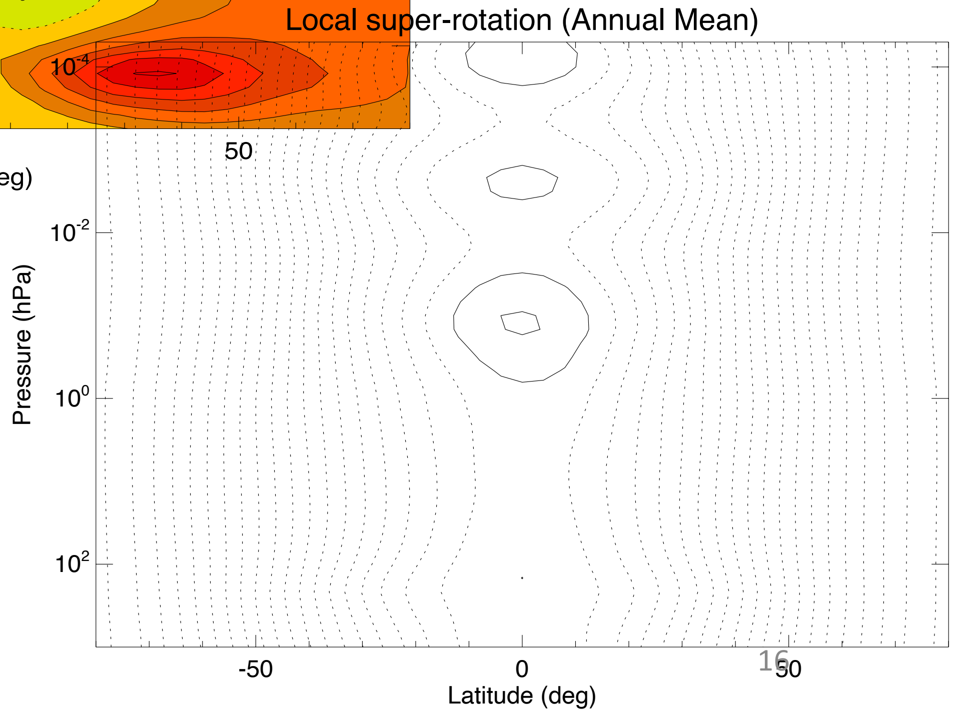
Ψ



Where do we observe super-rotation (and how strong)?

- Earth's troposphere
  - Equatorial values of *s < 0* most of the time
  - Mass-weighted (compressible) global values of *S* ~ 0.010-0.017
  - Magnitudes ~ 0.01
- Earth's stratosphere & Mesosphere
  - Annual mean equatorial $\bar{u}$ westward except around 0.1 and 10^{-3} hPa
  - NB Eddy driven!





Data credit: Randal et al [2004]

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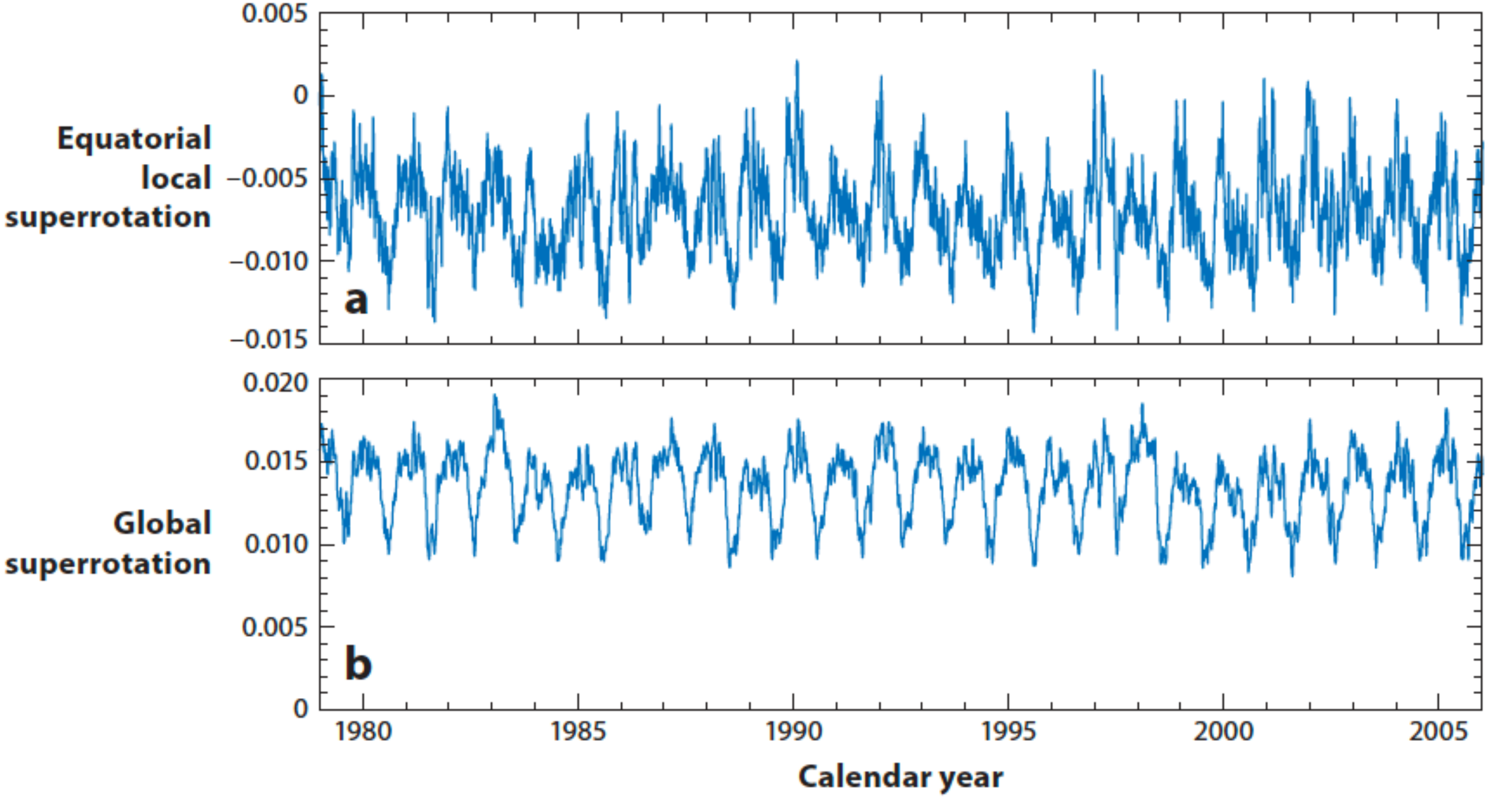
  - NB Eddy driven!



Credit: Read & Lebonnois (2018 Ann Rev. EPS)

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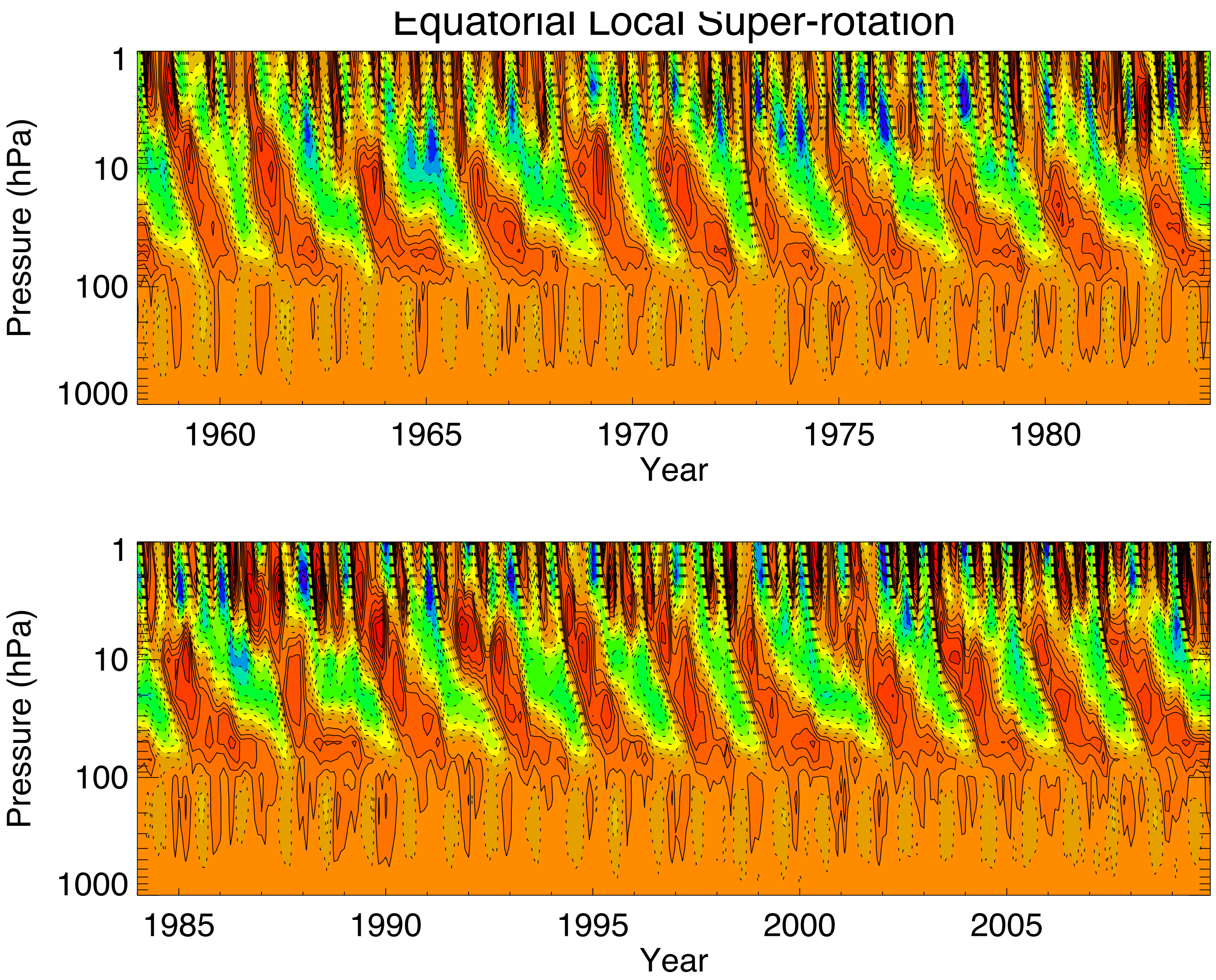

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  - NB Eddy driven!
- Earth's oceans!
  - Equatorial undercurrent….

Credit: Cornillon et al. [2019]

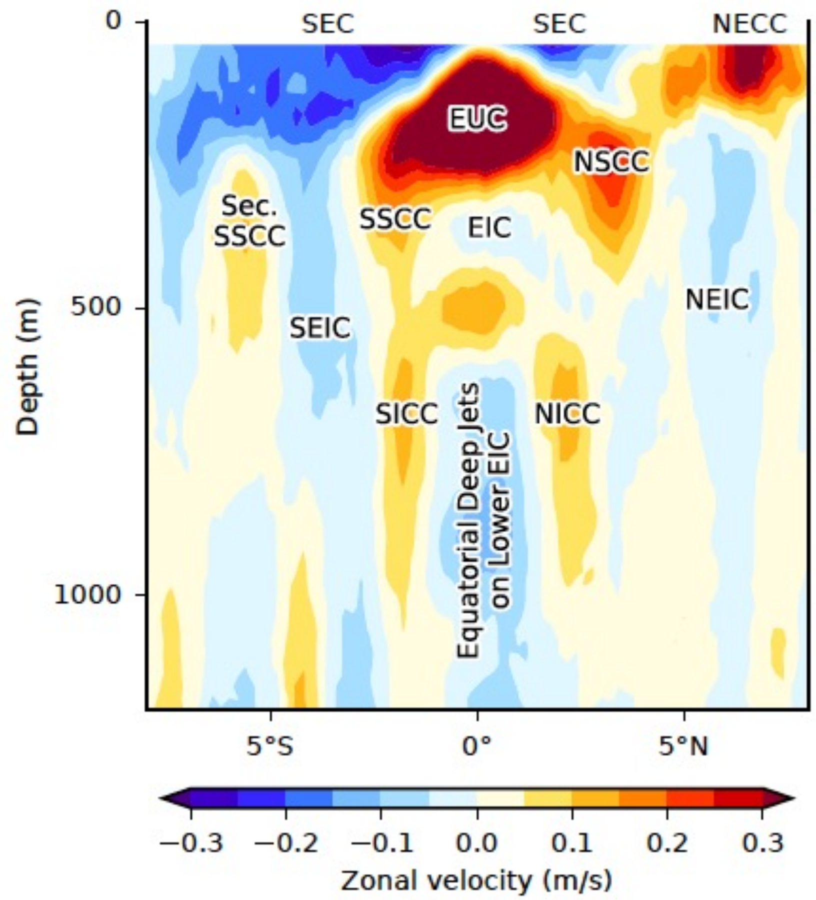
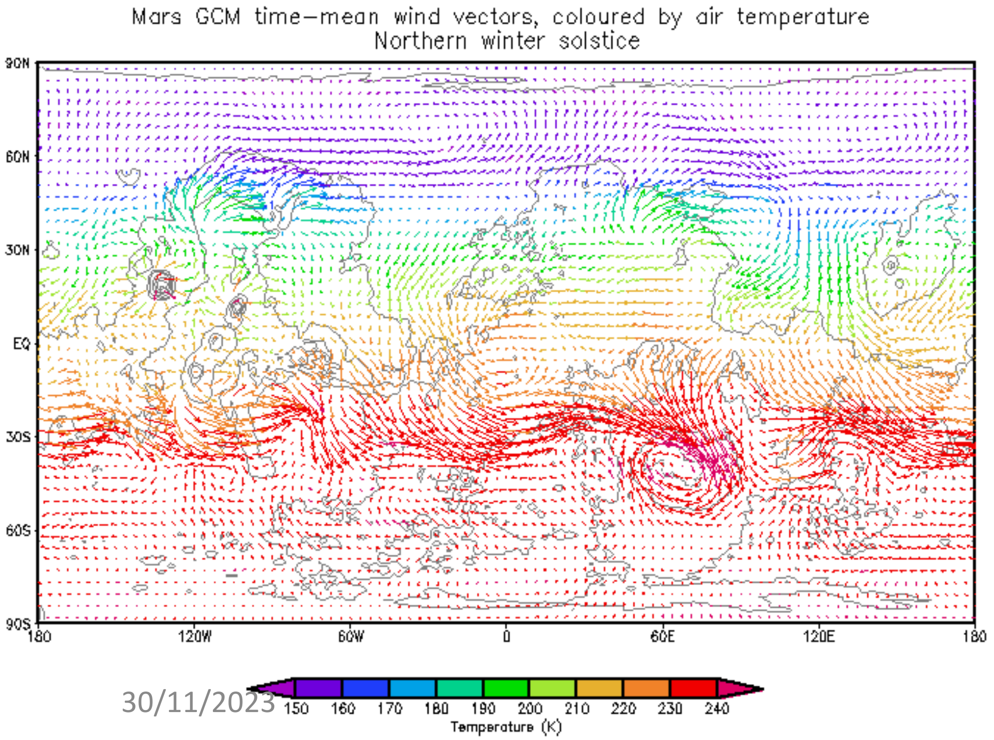


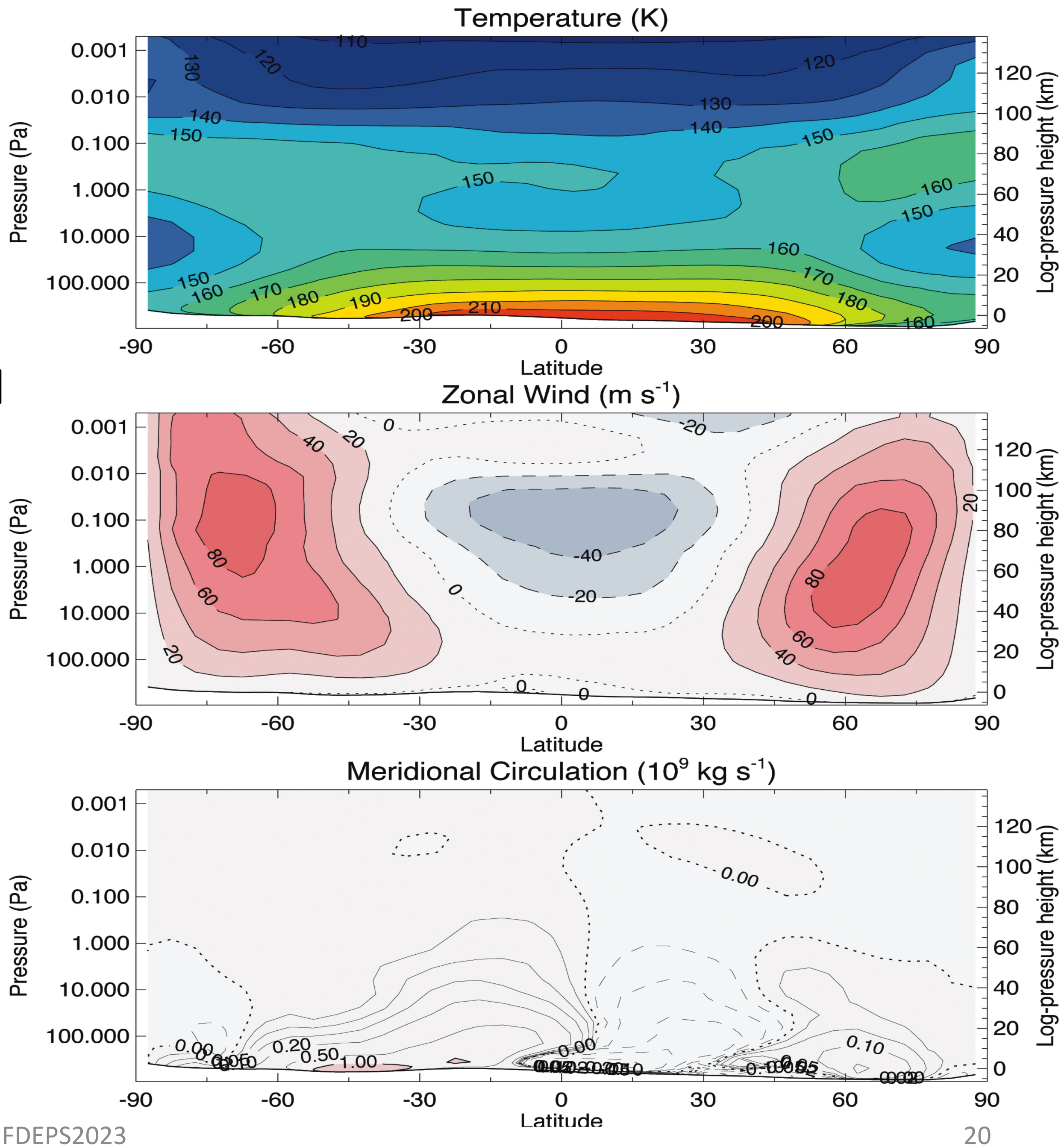
Figure 3.2 Mean zonal velocity from 9 shipboard ADCP sections that crossed the equator near 165°W. Three sections were made in 2004, one in 2007, two in 2008, two in 2010 and one in 2011. All measurements were made using a 38-kHz ADCP on the R/V Kilo Moana, which typically profiles to about 1200 m depth.

Atmosphere of Mars

Annual mean Climatology

[Read & Lewis 2004]





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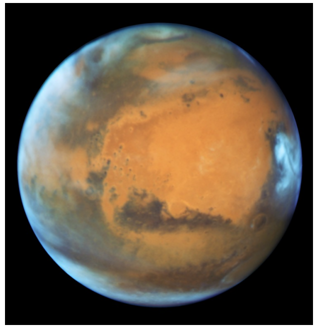
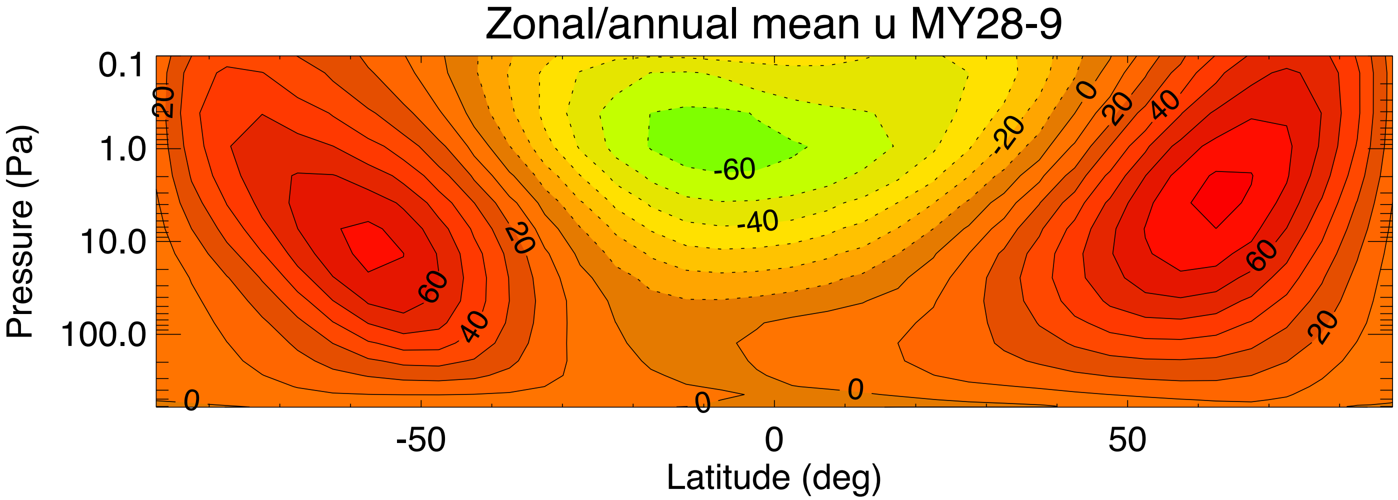
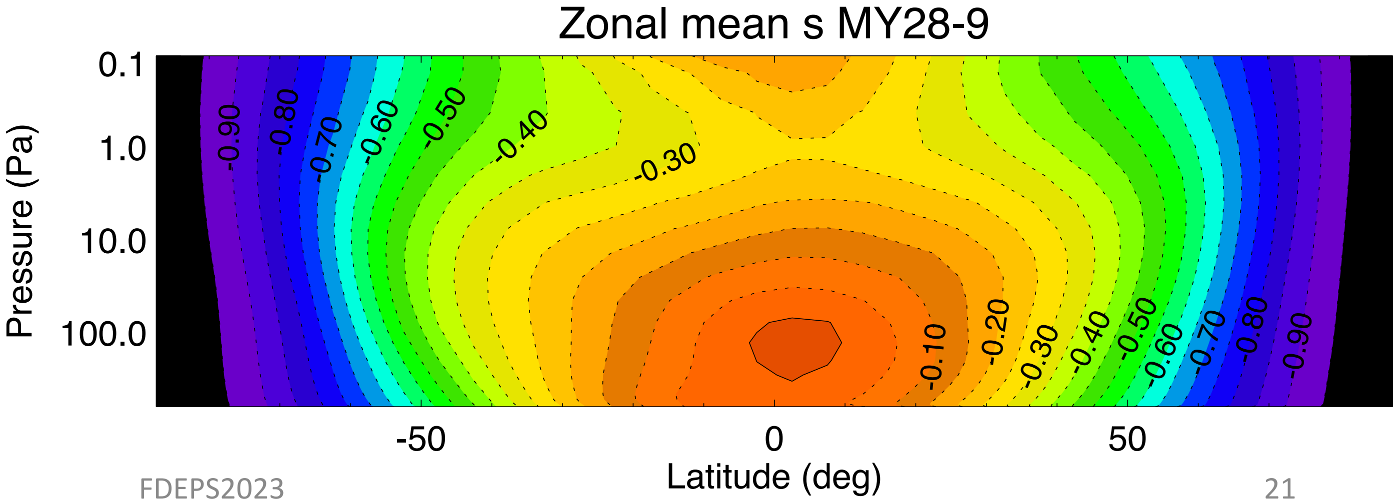


- Mars atmosphere
  - Annual mean zonal wind exhibits midlatitude westerly jets and mostly easterly flow in the tropics
  - Weak westerly flow on the equator below 20 km altitude
  - Local super-rotation s peaks at around 0.03
  - Seasonal variations?





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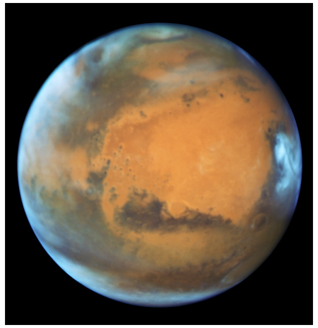
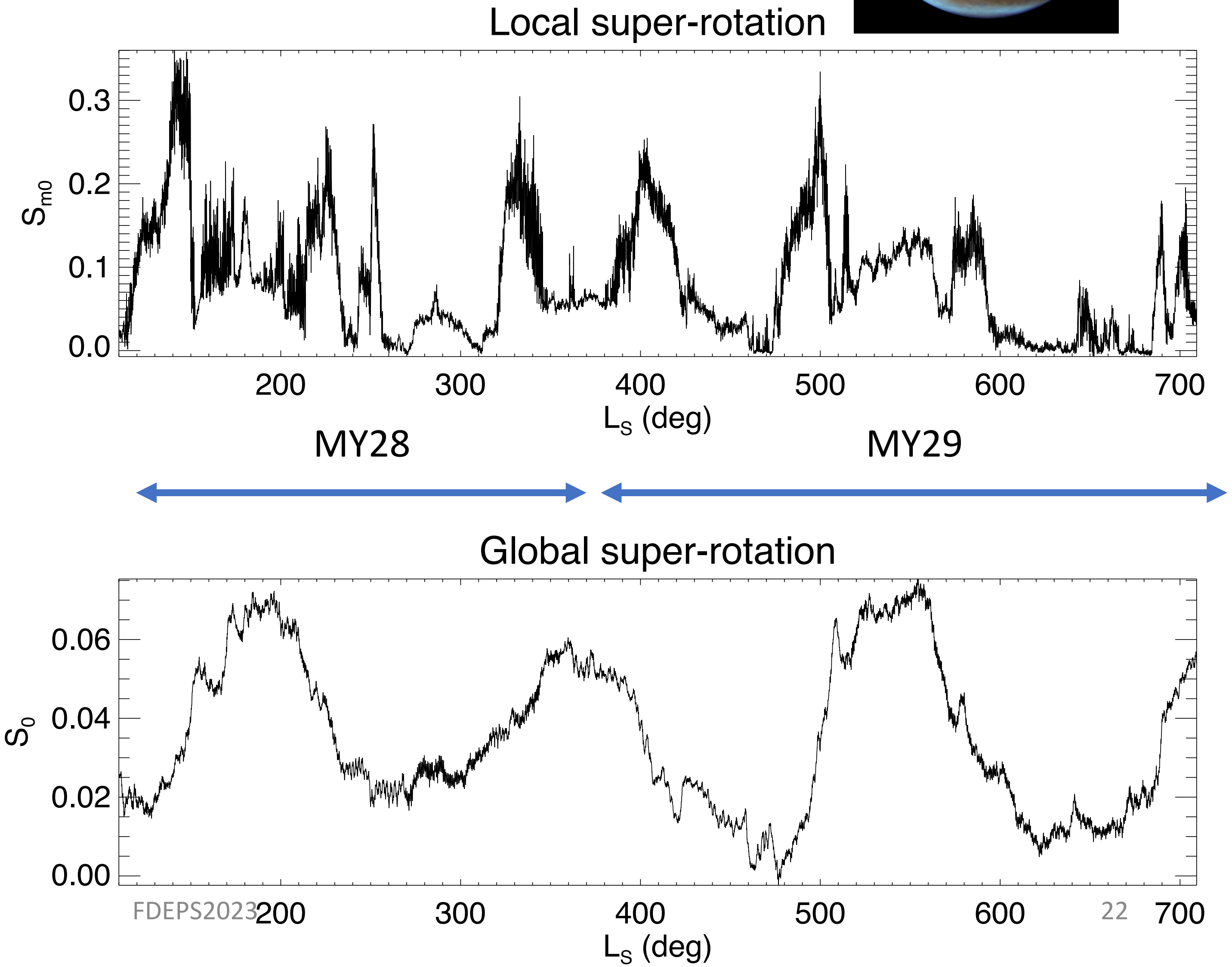


- Mars atmosphere
  - Annual mean zonal wind exhibits midlatitude westerly jets and mostly easterly flow in the tropics
  - Weak westerly flow on the equator below 20 km altitude
  - Local super-rotation s peaks at around 0.3
  - Seasonal variations?
  - Global super-rotation varies from ~0 to 0.07



Where do we observe super-rotation (and how strong)?

Cassini images: NASA/JPL

- Jupiter's atmosphere
  - Giant planet ($a \sim 11.a_{Earth}$) rotates rapidly (τ_{rot} = 9.926 hours)
  - Multiple eastward and westward zonal jets in each hemisphere at cloud tops
  - Strong (>100 m s^{-1}) eastward equatorial jet
  - Local super-rotation at cloud tops peaks at around s_{max} = +0.006 at the equator

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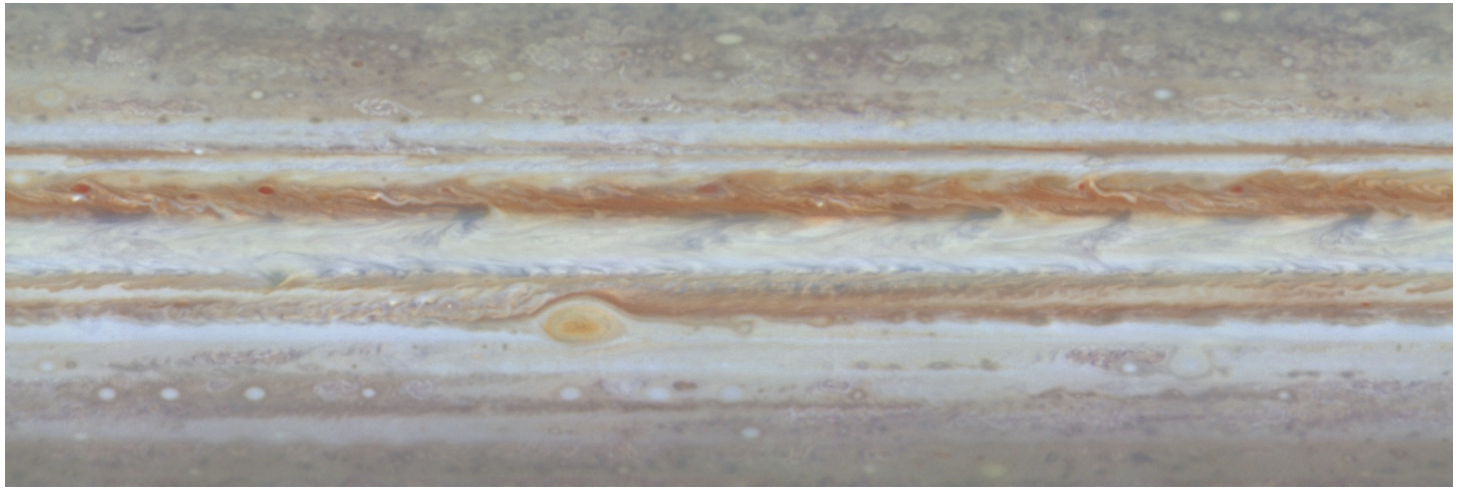

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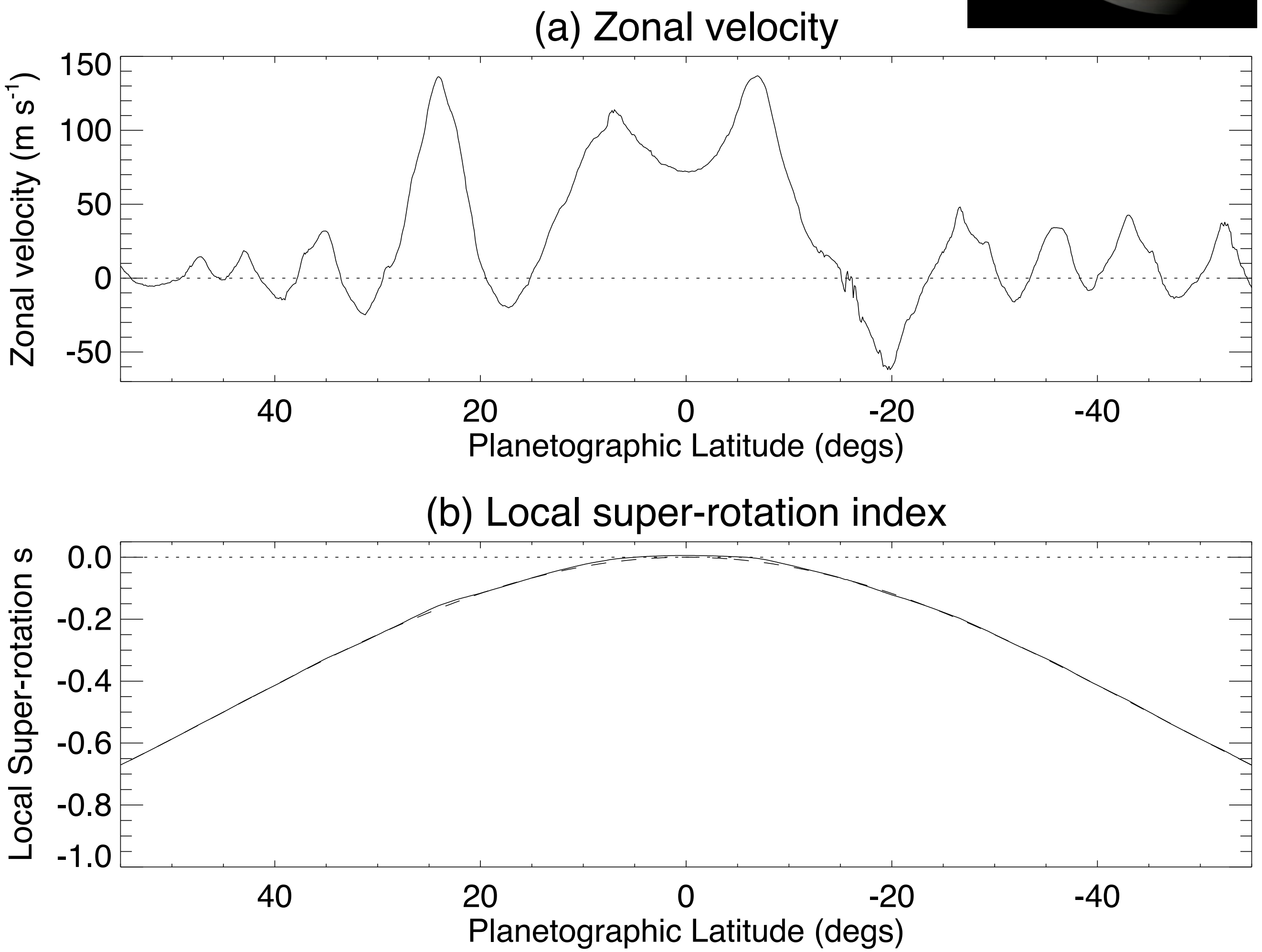

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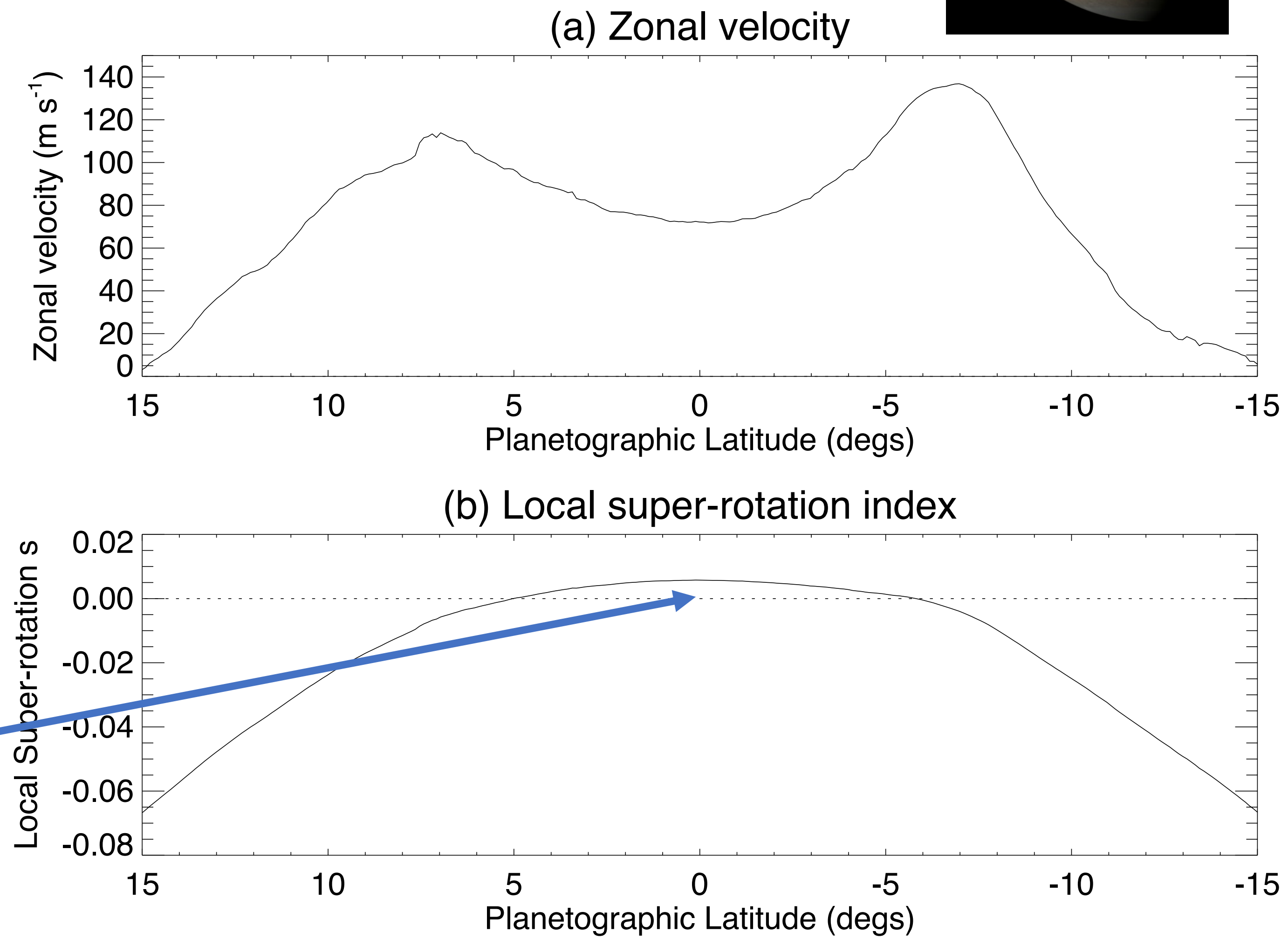

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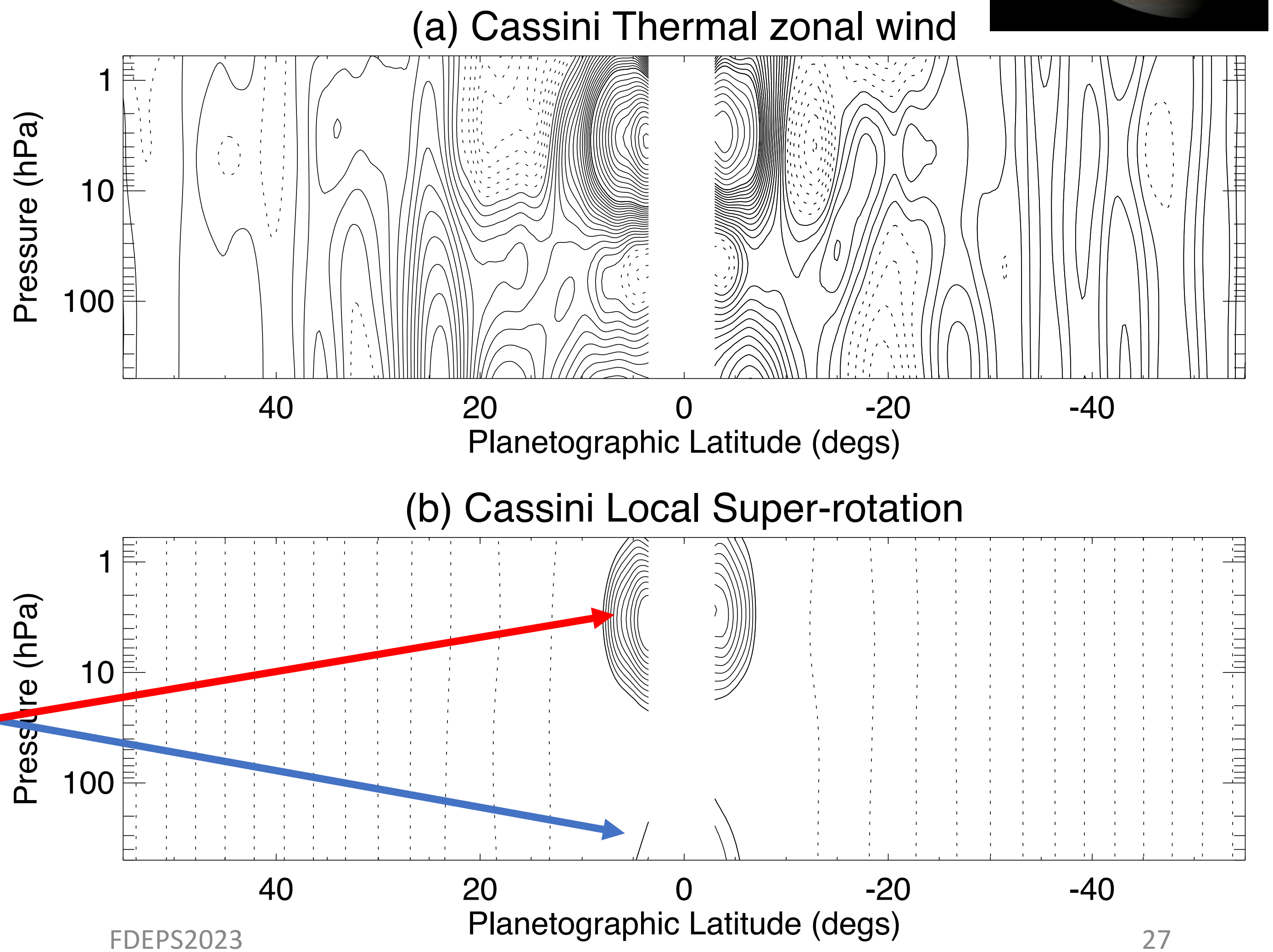
    - Up to s ~ 0.03 in the stratosphere?



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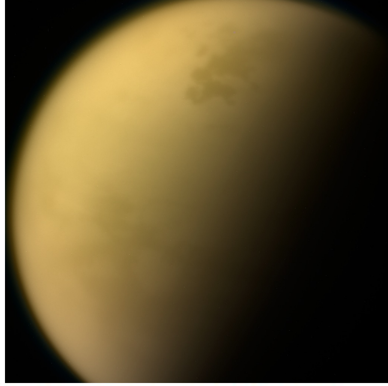
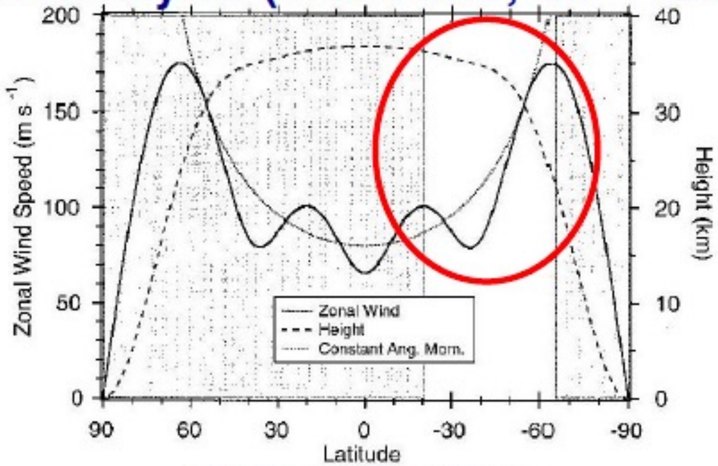
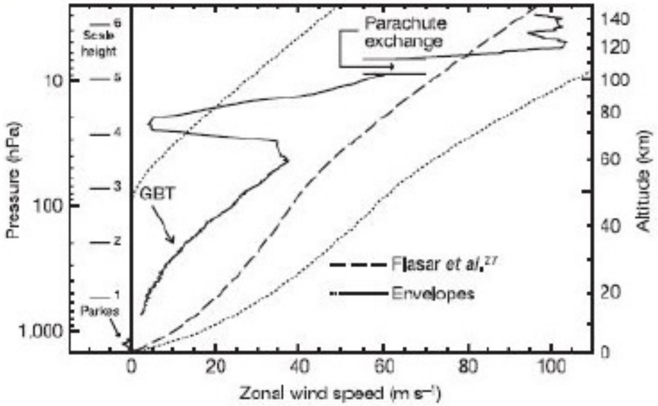


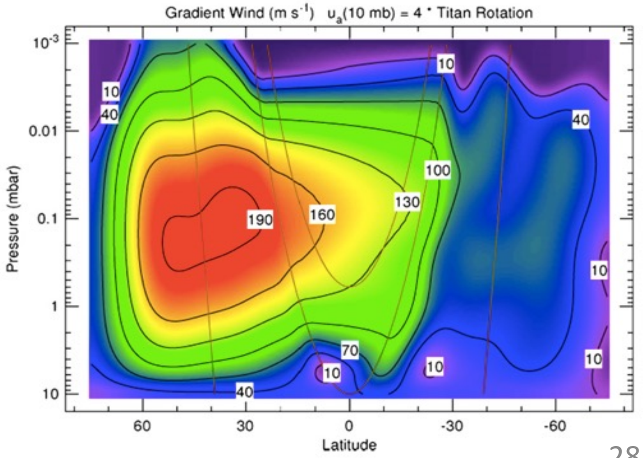
- Titan's atmosphere
  - Major satellite of Saturn
    - ($a \sim 0.4a_{Earth}$) rotates slowly (τ_{rot} = 15.95 days)
  - Substantial atmosphere rich in N_2 and CH_4
  - Strong (>100 m s^{-1}) eastward equatorial jet
    - Varies with season?

Stellar occultation analysis (0.25 mbar, Ls ~ 128°)



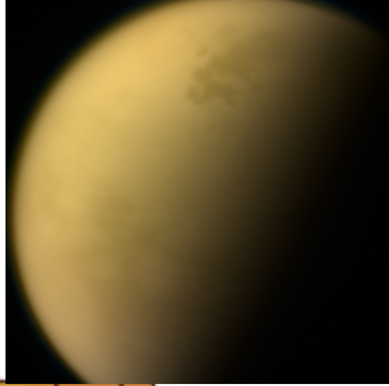
Huygens/DWE vertical profile at 10°S (Ls ~ 300°)





Cassini/CIRS thermal winds retrieval (Ls ~ 300°)

Where do we observe super-rotation (and how strong)?



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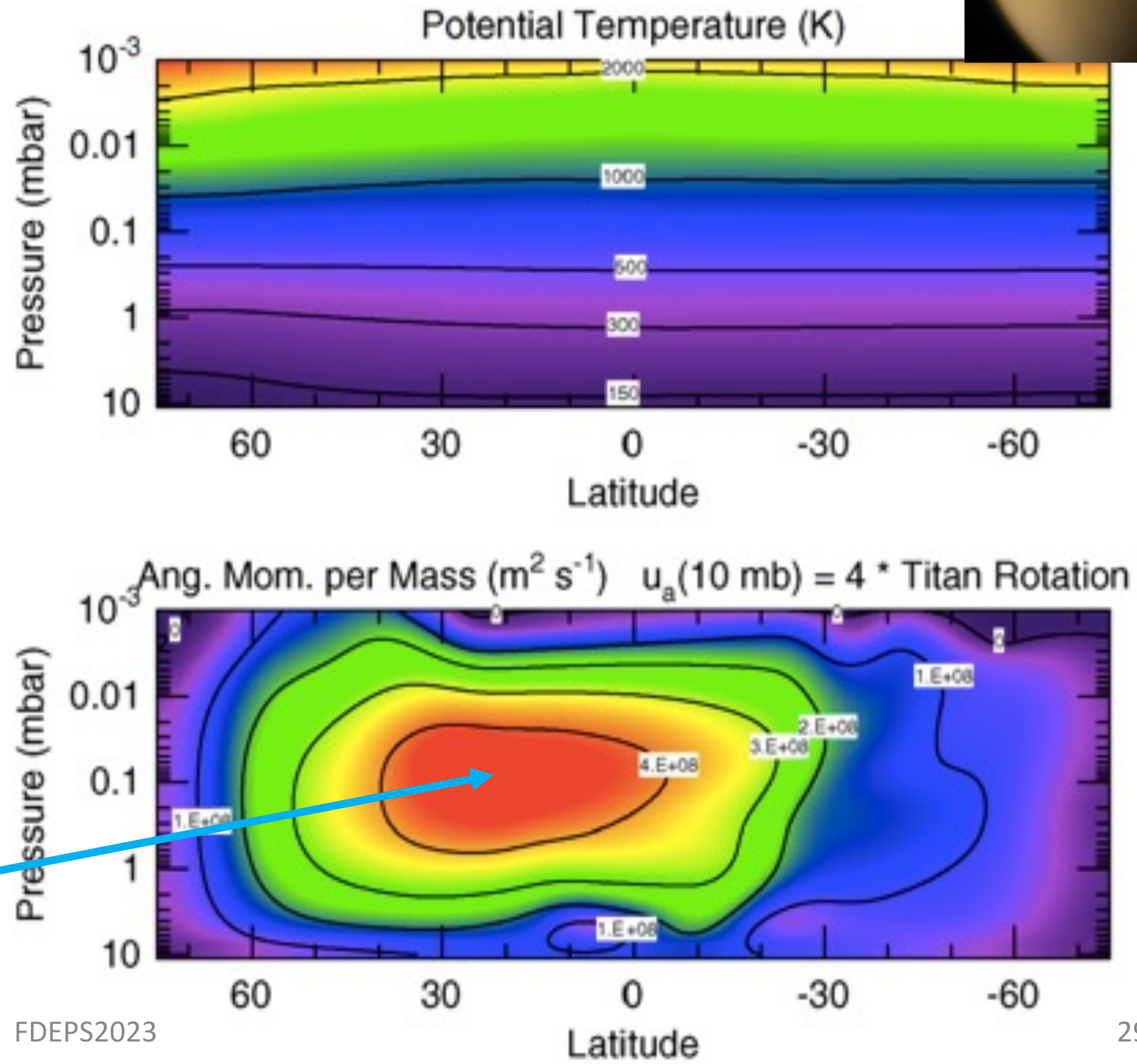
  - Local super-rotation at cloud tops peaks at around +15 at the equator

Cf $\Omega a^2 = 3.02 \times 10^7$ m^2 s^{-1}



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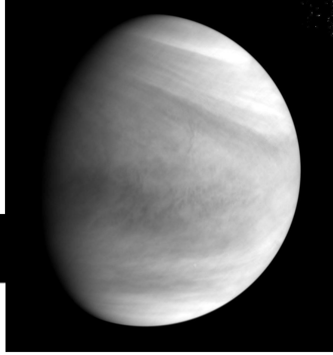
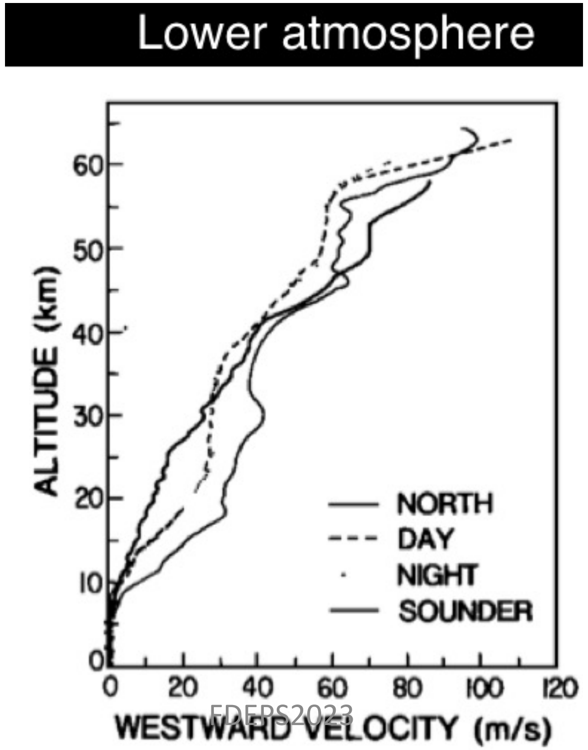
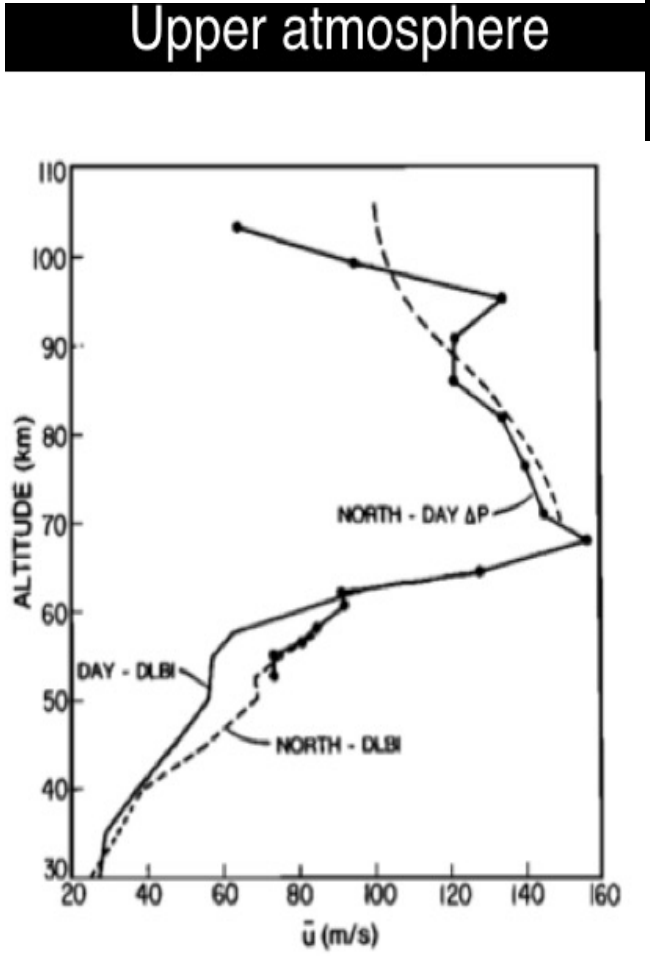


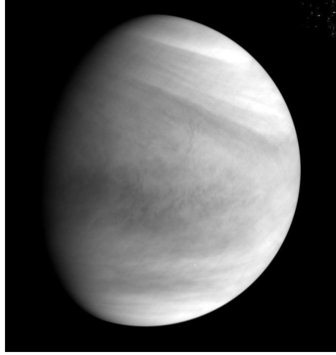
- Venus's atmosphere
  - Earth-sized planet ($a \sim 0.9a_{Earth}$) and rotates very slowly (τ_{rot} = 243 days ***retrograde***)
  - Massive atmosphere mainly of CO_2 with cloud decks of H_2SO_4 droplets around 40-60 km altitude
  - Strong (>100 m s^{-1}) eastward flow at cloud tops (peaks at around ~70 km altitude)
    - Weak prograde flow in the lower atmosphere





[Data from Pioneer Venus probes]

Where do we observe super-rotation (and how strong)?



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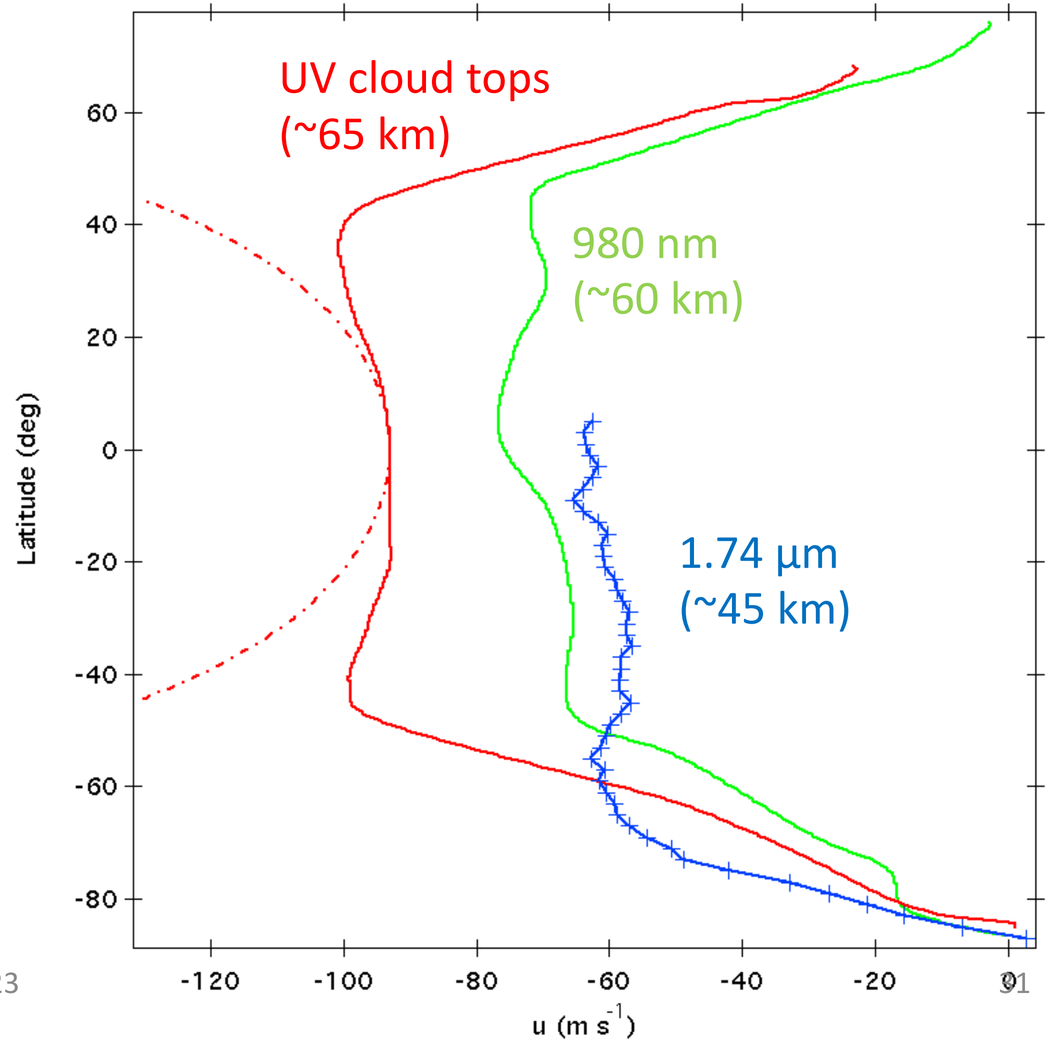
  - Strong (>100 m s^{-1}) eastward flow at cloud tops (peaks at around ~70 km altitude)
    - Weak prograde flow in the lower atmosphere
    - ~ uniform with latitude between ±50° - 60°



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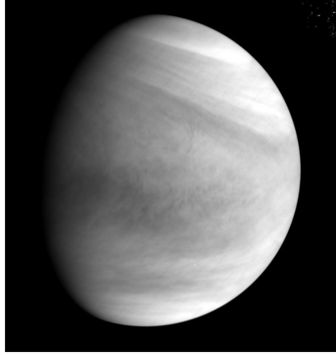

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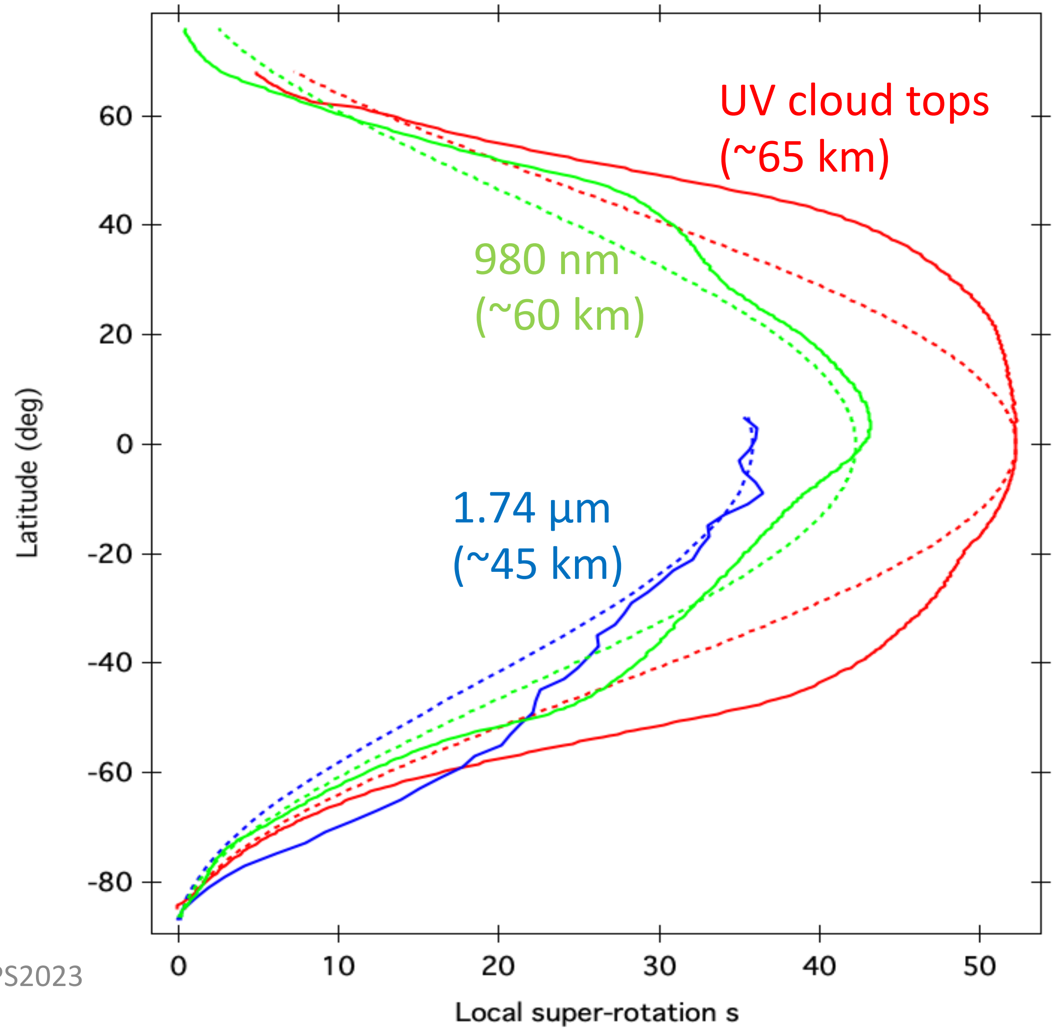
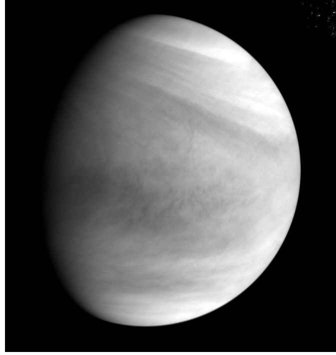
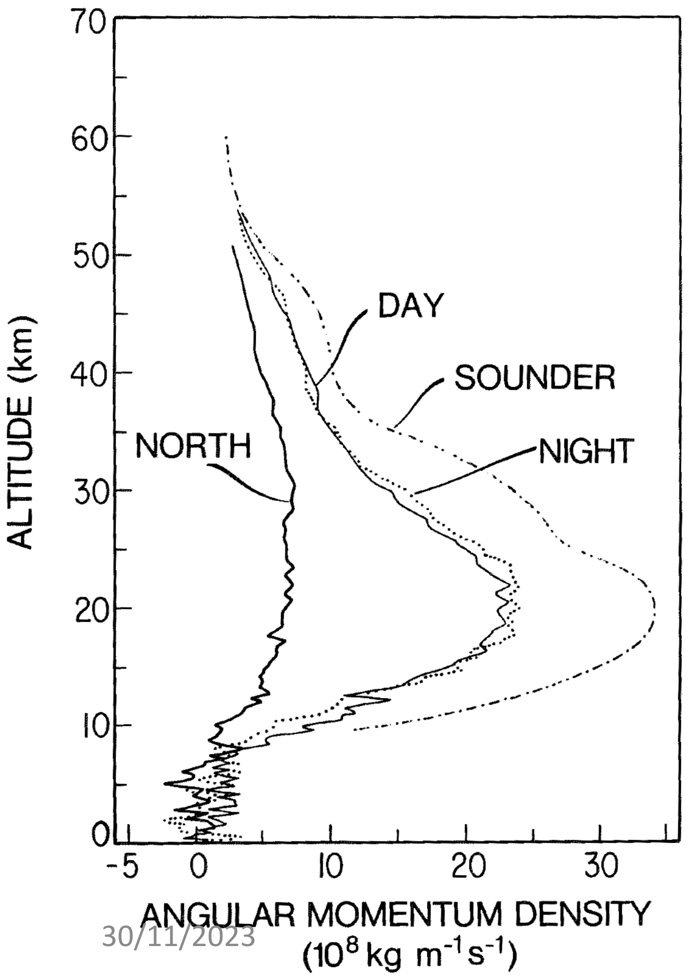
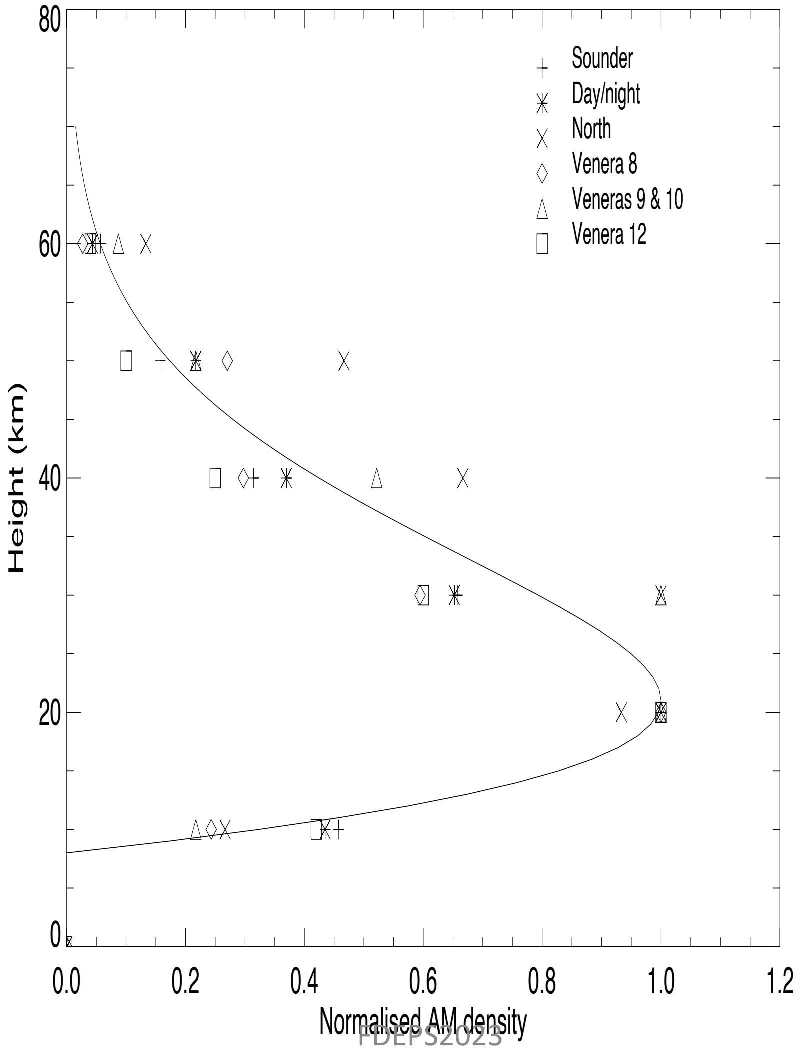
  - Strong (>100 m s^{-1}) eastward flow at cloud tops (~60 km altitude)
  - Local super-rotation at cloud tops peaks at around +52.2 at the equator
    - Decreasing towards the surface



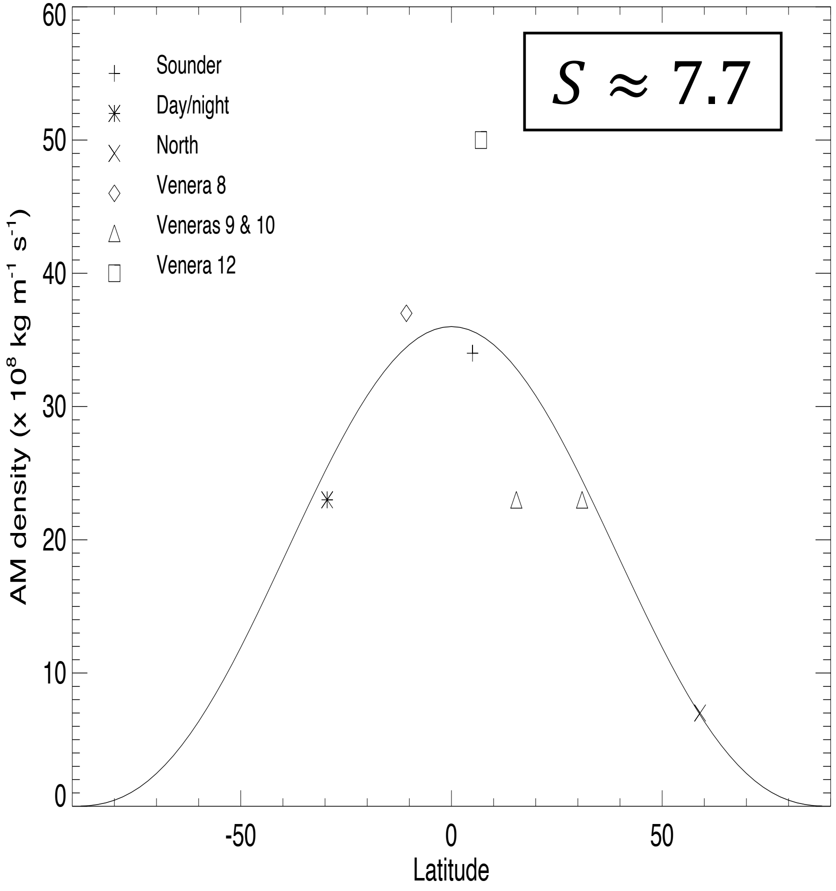
Where do we observe super-rotation (and how strong)?





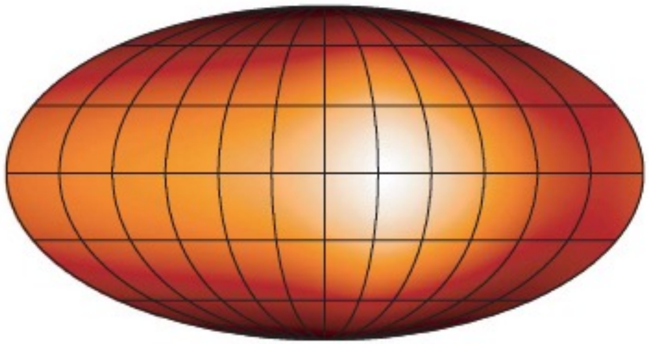


- Estimating global super-rotation

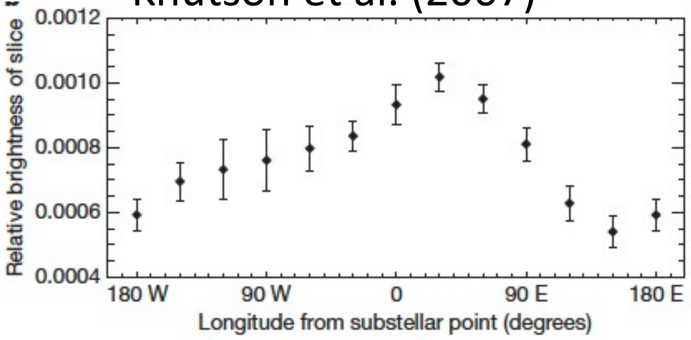


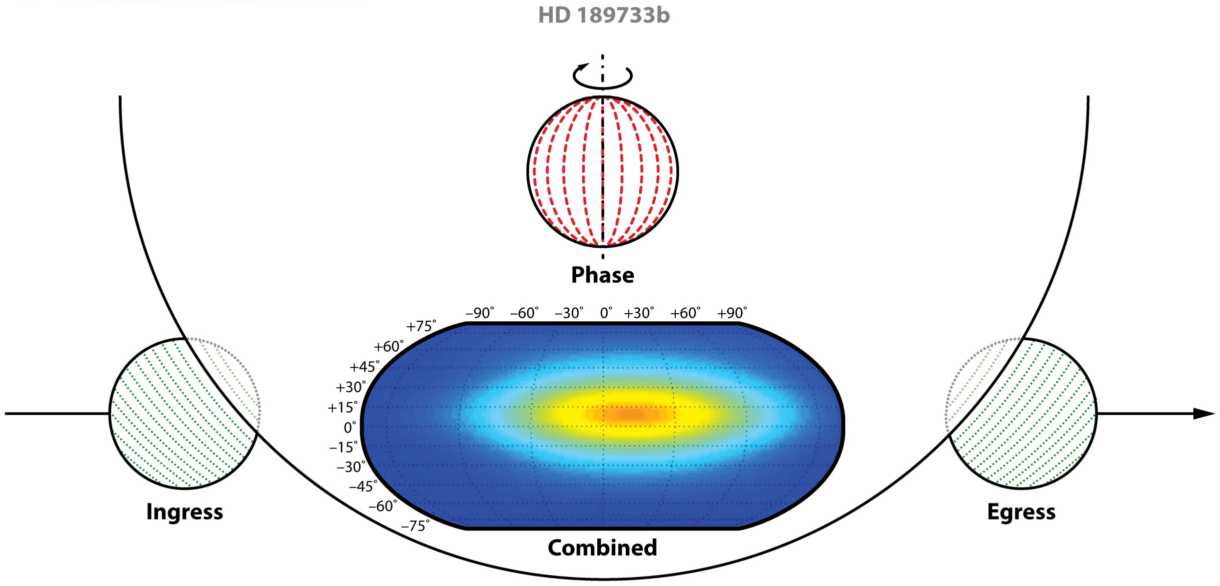
Where do we observe super-rotation (and how strong)?

- Hot Jupiter exoplanets
  - Jupiter-sized planets (a ~ $10a_{Earth}$) in orbits close to their parent star.
  - Rotation likely to be locked in 1:1 resonance with orbit, due to gravitational tides
  - Phase curves from secondary transits indicate eastward displacement of sub-stellar hot spot
  - Advected by strong (>1000 m s^{-1}) eastward (super-rotating) flow…?



Knutson et al. (2007)





Heng K, Showman AP. 2015.
Annu. Rev. Earth Planet. Sci. 43:509–40

Summary of super-rotating atmospheres

Planet	a (km)	Ω (s^{-1})	u_{max} (m s^{-1})	s_{max}	s_{Global}
Earth	6371	7.27×10^{-5}	30	-0.015 – +0.04	0.0135 ± 0.002
Mars	3396	7.09×10^{-5}	30	0.16	0.04
Jupiter	69911	1.76×10^{-4}	-60 - +140	0.005 – 0.016	-
Saturn	58232	1.64×10^{-4}	350 - 430	0.035 – 0.045	-
Uranus	25562	1.01×10^{-4}	-80	-0.03	-
Neptune	24622	1.08×10^{-4}	-400	-0.15	-
Pluto	1152	1.14×10^{-5}	10 - 15	0.76 – 1.1	-
Triton	1353	1.24×10^{-5}	5 - 10	0.3 – 0.6	-
HD189733b	79500	3.28×10^{-5}	2400	0.93	-
HD209458b	94380	2.06×10^{-5}	1940	1.00	-
Titan	2576	4.56×10^{-6}	100 - 180	8.5 - 15	2
Venus	6052	2.99×10^{-7}	100 - 120	50 - 60	7.7

Super-rotation with eddies: The Hide/Starr theorem (II)

- For non-axisymmetric flows we need to take account of processes which break AM conservation
  - [Molecular viscosity?]
  - Eddies….

  and look for constraints on eddy

  transport of AM
- Consider the zonally averaged form of (6) [or (7)] in a ***steady state***

$$\nabla.(\overline{\boldsymbol{u}m}) = \overline{F_B}a\cos\varphi = \overline{\mathcal{F}_B} \qquad (8)$$

Victor Starr [1909 – 1976]

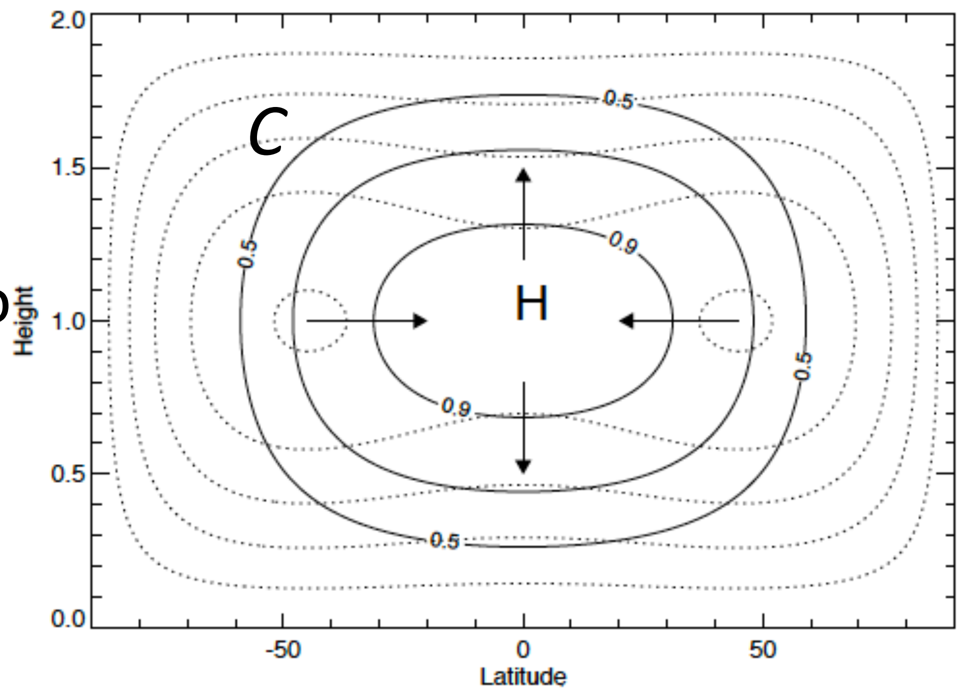
Super-rotation with eddies: The Hide-Starr theorem (II)

- Now integrate (8) over the toroidal annular volume or ring of fluid enclosed by a closed contour *C* in the meridional plane
- If there exists a local maximum of $\overline{m} = m_0$ at point H in the meridional plane, we can take *C* to be a contour of constant $\overline{m} = m_0 \ - \delta$
- LHS of the integral of (8) over volume *C*
- $\iiint_C \nabla.(\overline{\boldsymbol{u}}\overline{m})dV = \iint_C \overline{m}\overline{\boldsymbol{u}}.d\boldsymbol{n}$ (9a)
  - [Divergence theorem]

$$= (m_0 - \delta) \iint_C \boldsymbol{u}.d\boldsymbol{n} \quad (9b)$$

$$= 0 \quad (9c)$$

  - by mass conservation [NB for steady state only!]



Hide-Starr theorem (II): constraints on eddies

- Now consider the RHS of (8). Suppose we can represent the torque due to eddies by

$$\overline{\mathcal{F}_B} = \nabla.\boldsymbol{E} \qquad (10)$$

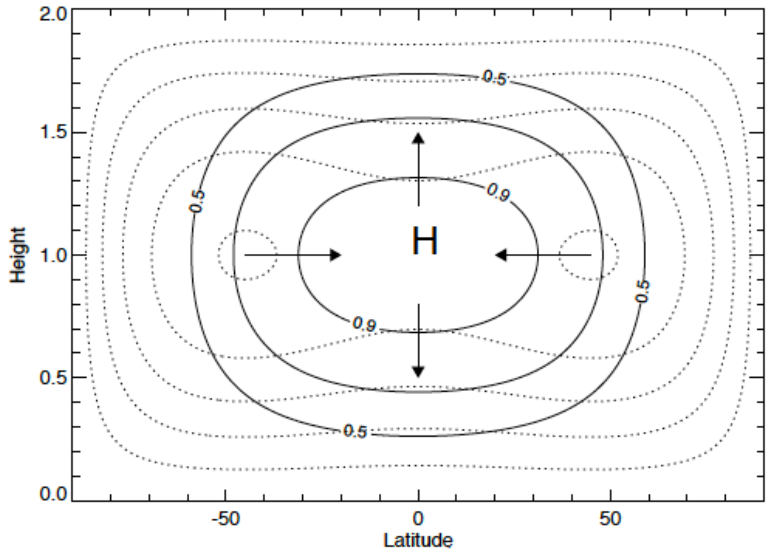
where $\boldsymbol{E}$ is a flux of AM due to eddies

- From (9b) and (9c), therefore

$$\iiint_C \mathcal{F}_B\, dV = \iint_C \boldsymbol{E}.d\boldsymbol{n} = 0 \qquad (11)$$

- This can only be satisfied if $\boldsymbol{E}.\boldsymbol{\nabla}\overline{m}$ takes either sign [or $\boldsymbol{E}$ is everywhere parallel to contours of $\overline{m}$], so in general…

- Eddy angular momentum fluxes **E** must be able to transfer $\overline{m}$ up-gradient as well as down-gradient



Hide-Starr Theorem(s): Summary

- We can define specific (axial) angular momentum

$$m = (\Omega a \cos\varphi + u) a \cos\varphi$$

  - Where a = planetary radius; Ω = rotation rate, u = zonal velocity & ϕ = latitude
- In zonal mean:

$$\frac{\partial \bar{m}}{\partial t} + \nabla.(\bar{m}\bar{\boldsymbol{u}}^*) = \nabla.\boldsymbol{E} + F$$

mean advection | eddy stress | body forces/torque

[***E*** is ~Eliassen-Palm flux
u* is TEM meridional velocity – see later]

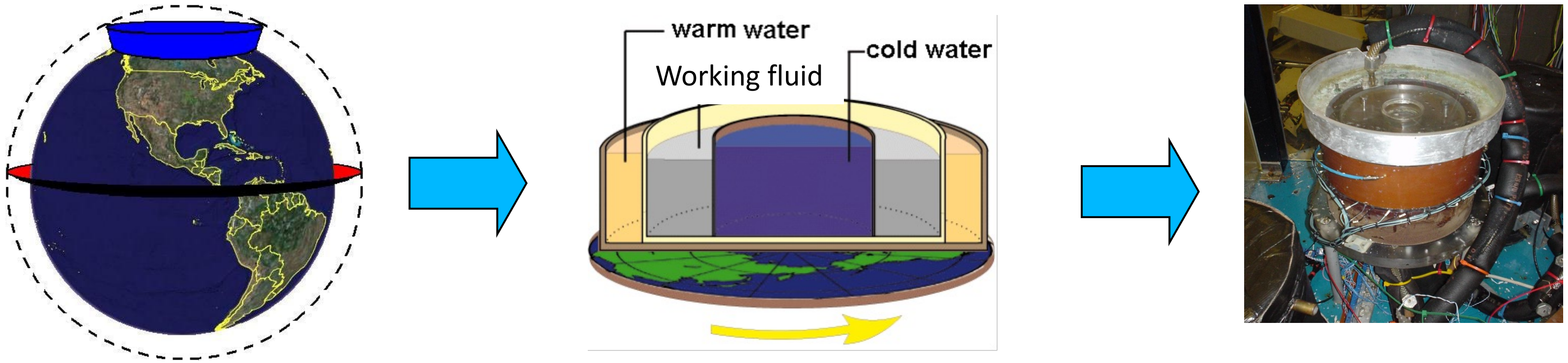
- NB *m* materially and globally conserved in frictionless, axisymmetric flows (***E**, F = 0*)
- HENCE
  - Equatorial local super-rotation is impossible in purely axisymmetric, **inviscid** flow
  - Local or global super-rotation must involve the existence of non-axisymmetric eddies

AND

  - Eddy angular momentum fluxes **E** must be able to transfer $\bar{m}$ up-gradient

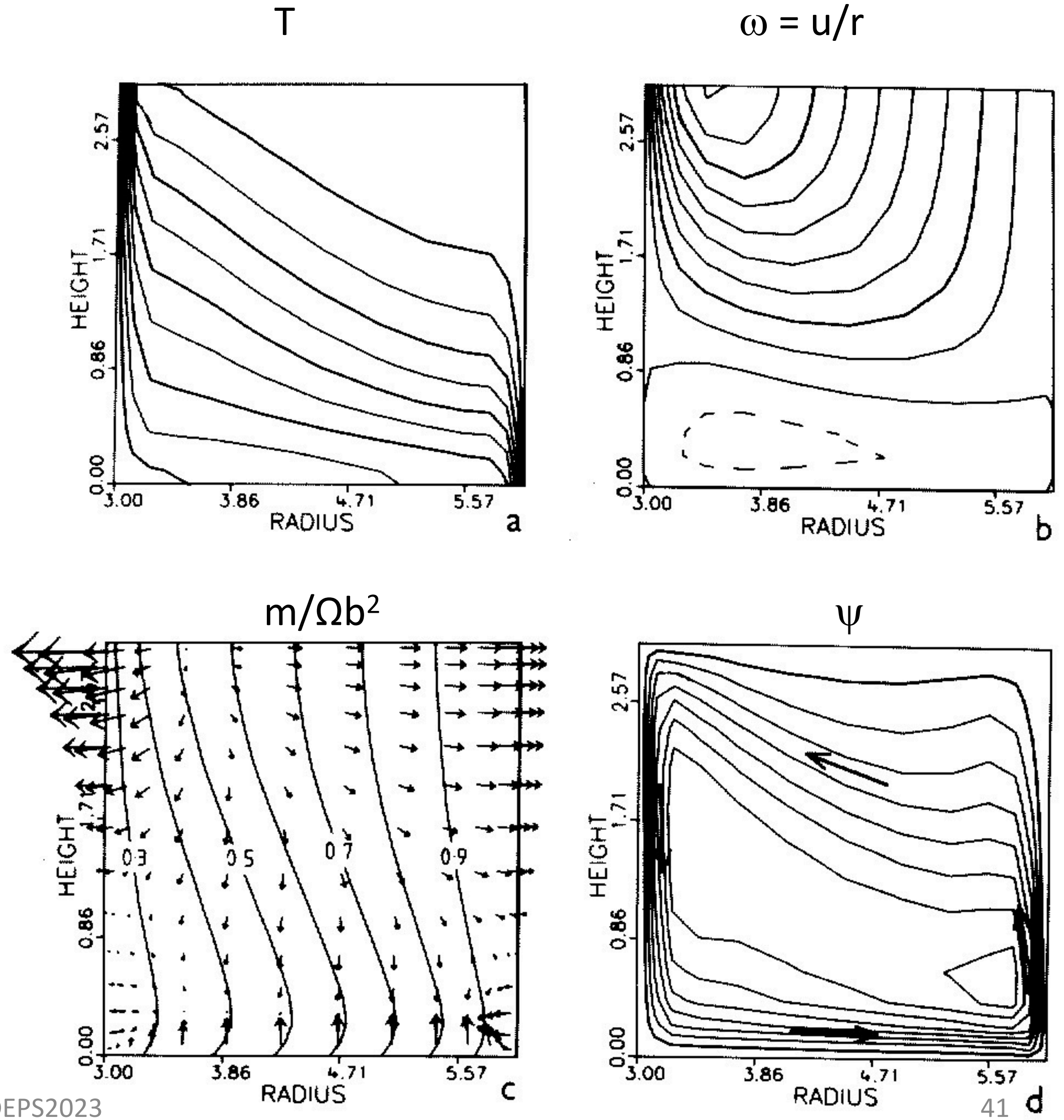
Effects of viscosity?

- What role could fluid viscosity play in super-rotation?
  - Analogous to “eddy viscosity”?
- Laboratory analogue of atmospheric circulation
  - Cylindrical geometry
  - Differentially heated rotating annulus



Effects of viscosity?

- Numerical simulation of axisymmetric flow in cylindrical annulus
- Heated outer cylinder and cooled inner
- Stress-free top surface and rigid, non-slip sidewalls
- Meridional overturning circulation (thermally direct)
- Super-rotation is very weak
  - s < 0.01
  - S ~ 0.08



Effects of viscosity?

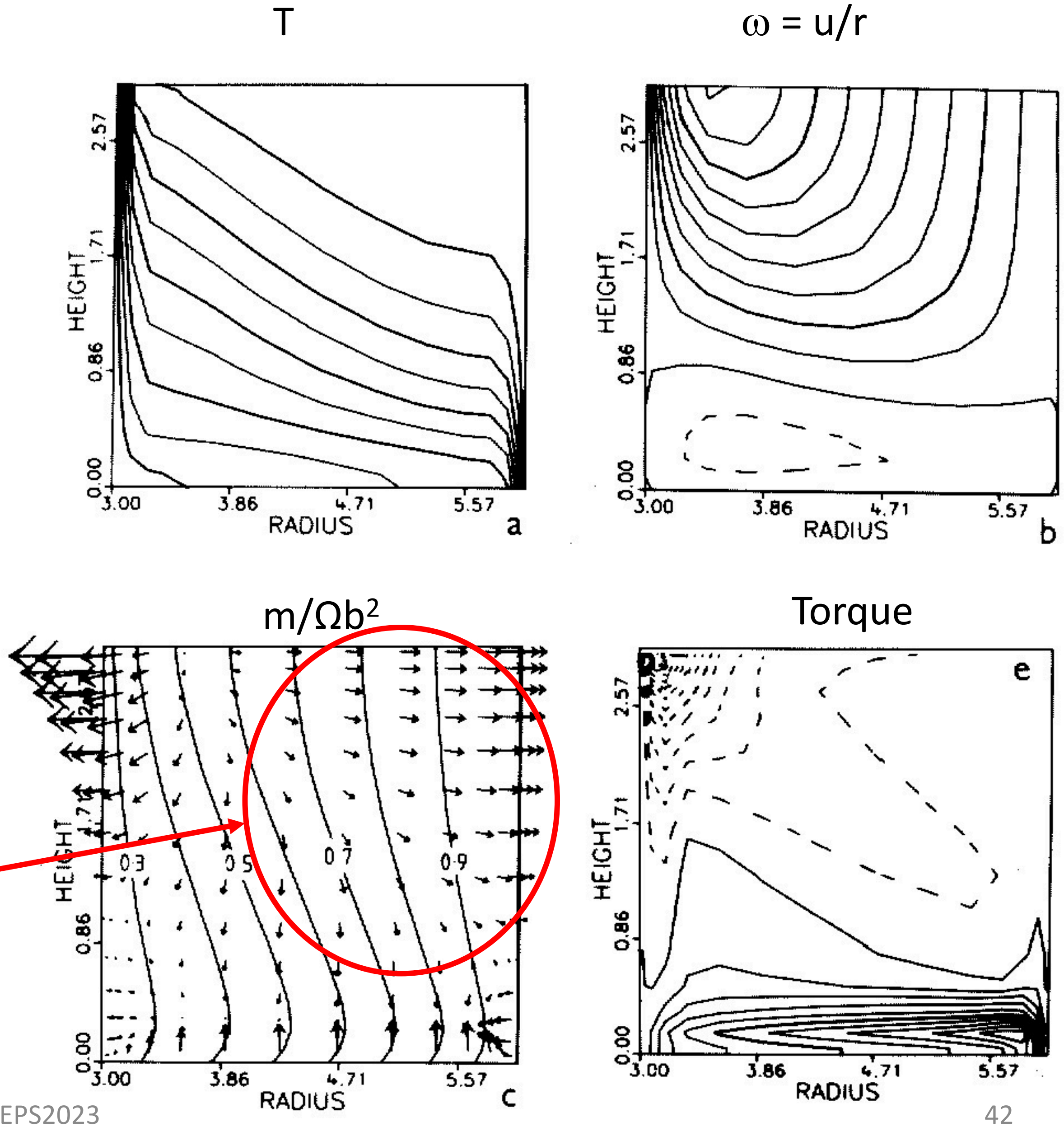
- The viscous term in (6) or (7) can be written as

$$F_B = -\nabla . \boldsymbol{F}$$

- Where

$$\boldsymbol{F} = -\nu r^2 \nabla \omega$$
$$= -\nu r^2 \nabla \left(\frac{m}{r^2}\right)$$

- Viscous forces/torques act to transfer AM downgradient for **angular velocity**
  - i.e. to relieve tangential stress
- Note that ***F*** satisfies (11) in acting up-gradient for *m* horizontally -
  - **but is downgradient for *m* vertically**



Viscous AM fluxes – up- or down-gradient?

$$\boldsymbol{F} = -\nu r^2 \nabla\omega = \nu r^2 \nabla\left(\frac{m}{r^2}\right) \text{ and}$$
$$m = r(u + \Omega r)$$

- So projection of ***F*** onto ∇m determines direction of transport

$$\boldsymbol{F}.\nabla m = -\nu\left[|\nabla m|^2 - \frac{1}{r}\frac{\partial}{\partial r}(m^2)\right]$$

- Hence
  - $\boldsymbol{F}.\nabla m < 0$ if ***F*** is down-gradient for *m*
  - $\boldsymbol{F}.\nabla m > 0$ if ***F*** is up-gradient for *m*
- Also note that $\frac{\partial m}{\partial z}$ is parallel to $\frac{\partial \omega}{\partial z}$
  - so the vertical component of ***F*** is always downgradient

Effects of viscosity?

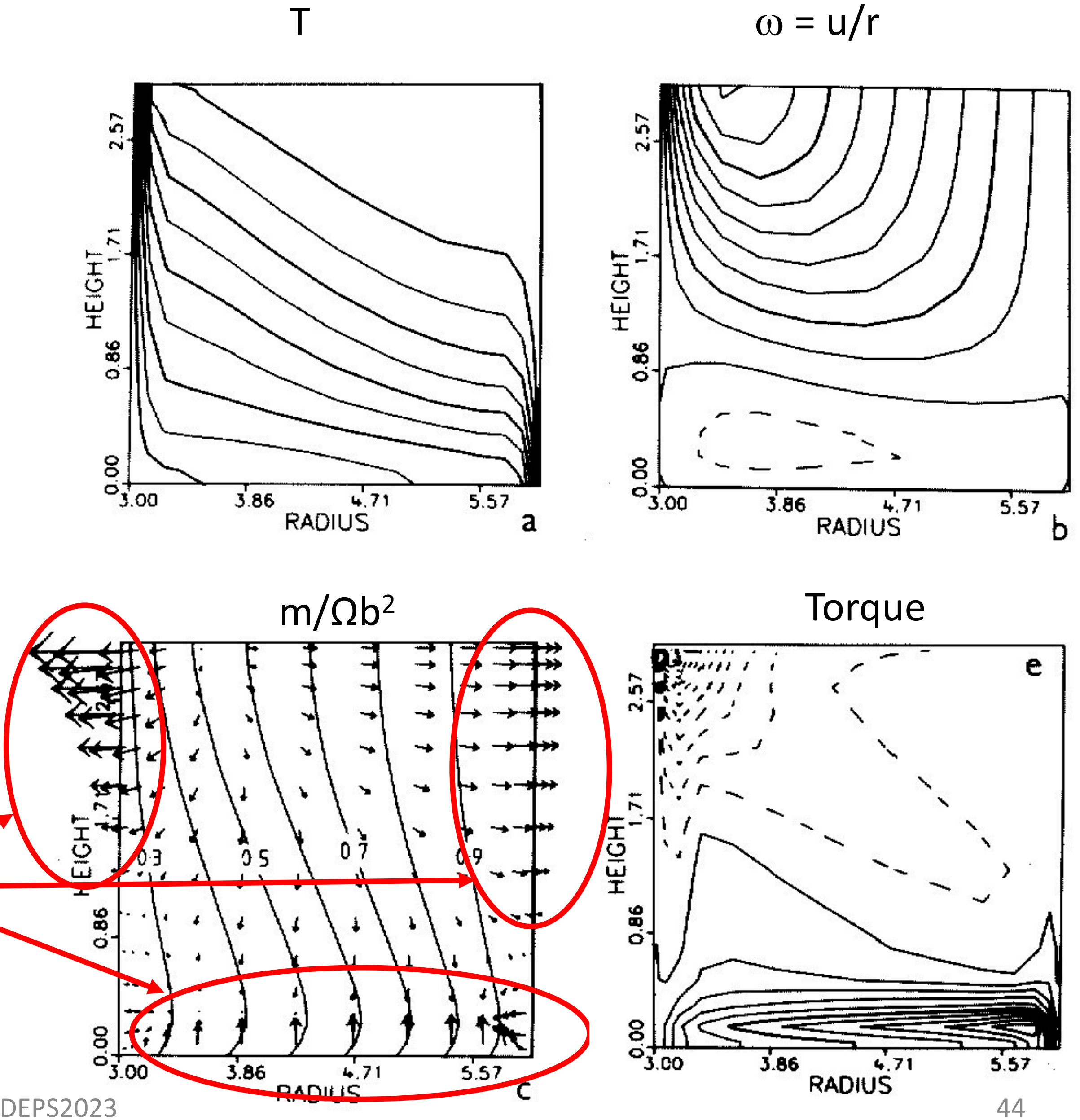
- The viscous term in (6) or (7) can be written as

$$F_B = -\nabla . \boldsymbol{F}$$

- Where

$$\boldsymbol{F} = -\nu r^2 \nabla \left(\frac{m}{r^2}\right)$$

- This allows AM to enter or leave the flow through boundaries that are not stress-free
- Allows the flow to gain or lose AM and change the global super-rotation
  - $S = 0.08$ and $s_{max} \sim 0.01$ for this case



Effects of viscosity?

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- Where

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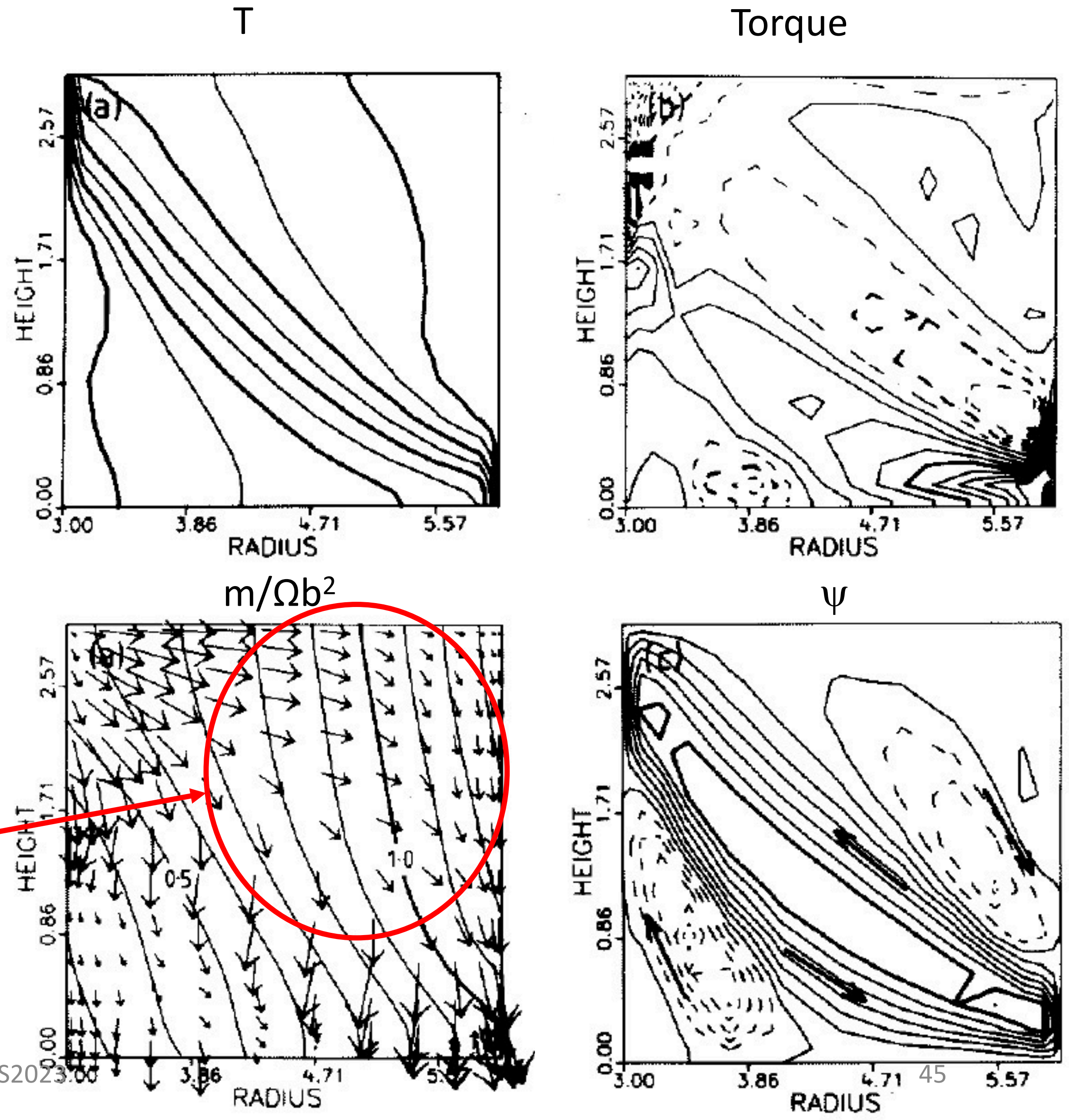
- This allows AM to enter or leave the flow through boundaries that are not stress-free
- Note that ***F*** satisfies (11) in acting up-gradient for *m* horizontally
- Much larger super-rotation with stress-free side boundaries
  - s_{max} ~ 0.35
  - S ~ 0.36

Now with stress-free side boundaries



Effects of viscosity?

- The viscous term in (6) or (7) can be written as

$$F_B = -\nabla.\boldsymbol{F}$$

- Where

$$\boldsymbol{F} = -\nu r^2 \nabla\left(\frac{m}{r^2}\right)$$

- This allows AM to enter or leave the flow through boundaries that are not stress-free
- Note that ***F*** satisfies (11) in acting up-gradient for *m* horizontally
- Much larger super-rotation with stress-free side boundaries
  - $s_{max} \sim 0.35$
  - $S \sim 0.36$

[Read 1986]

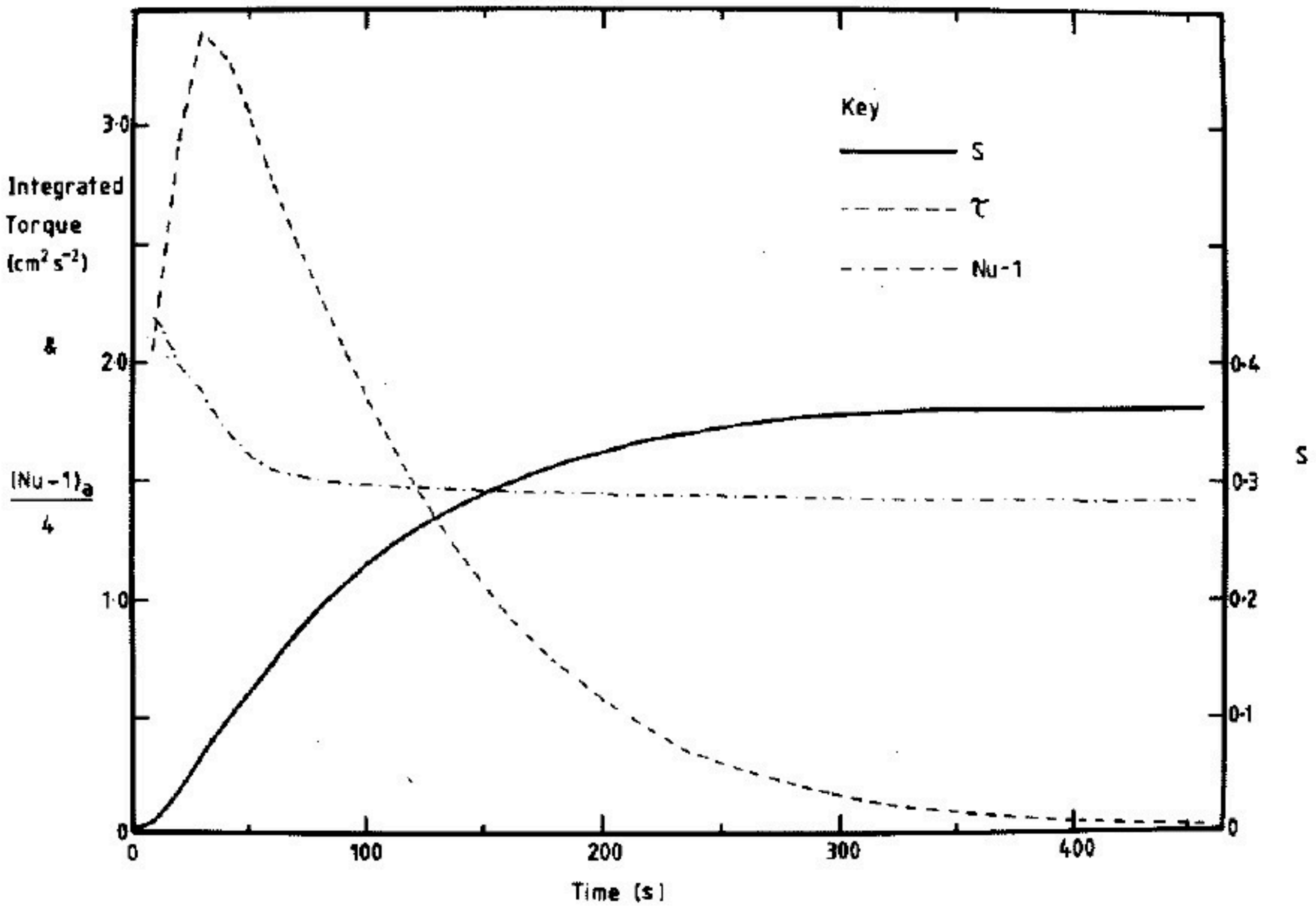
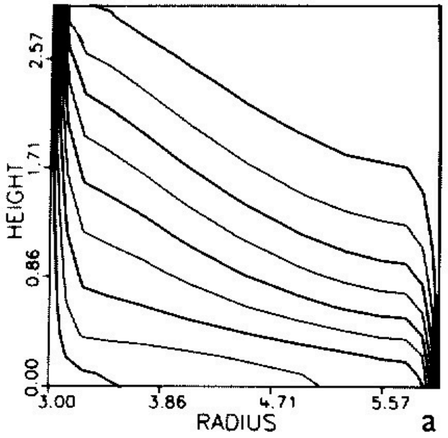


Figure 4. The evolution of various integrated properties of the flow in case B (see text) for the first 450 s of model time after initialization: (i) global super-rotation parameter S; (ii) total torque τ; (iii) Nusselt number Nu_a (measured at the inner sidewall).

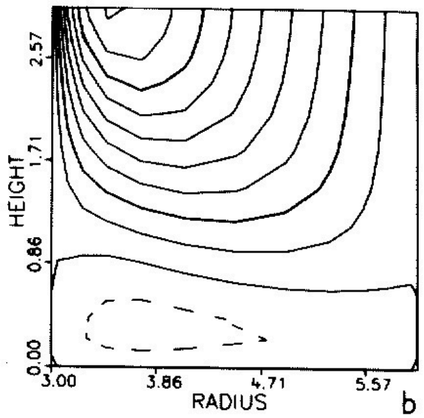
How to predict the magnitude of *S*?

A scaling theory for super-rotation in a rotating annulus

[Read1986]



- Governing equations for steady axisymmetric flow in cylindrical annular cavity



$$\nu(\nabla^2 v - v/r^2) = (1/r)\{\chi_z(v_r + v/r) - \chi_r v_z\} + (f_0/r)\chi_z \qquad (12)$$

where $f_0 = 2\Omega$ and subscripts again denote differentiation; an azimuthal vorticity equation,

$$\nu(\nabla^2 \zeta - \zeta/r^2) = J(\chi, \zeta/r) + g\alpha T_r - f_0 v_z - (1/r)(v^2)_z \qquad (13)$$

where α is the coefficient of cubical expansion, g is the acceleration due to gravity and ζ is the vorticity:

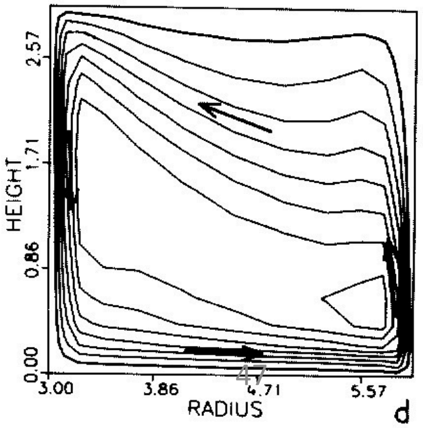
$$\zeta = u_r - w_z = (1/r)(\chi_{rr} - \chi_r/r + \chi_{zz}); \qquad (14)$$

and a thermodynamic equation

$$\kappa \nabla^2 T = (1/r)J(\chi, T). \qquad (15)$$

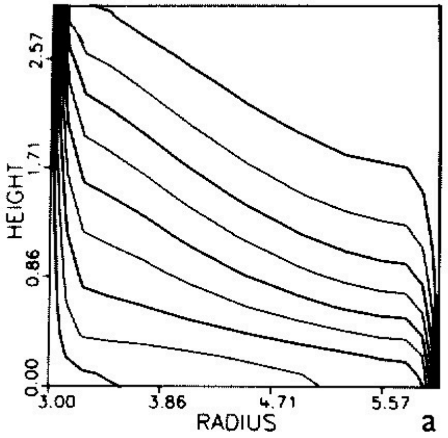
κ is the thermal diffusivity, and χ is the streamfunction for the meridional flow, so that

$$u = (1/r)\chi_z, \qquad w = -(1/r)\chi_r \qquad (16)$$



A scaling theory for super-rotation in a rotating annulus

[Read1986]



The problem may then be described fully by five dimensionless parameters, namely a Rayleigh number

$$Ra = g\alpha\,\Delta T\,L^3/\kappa\nu \tag{19}$$

where $L = (b - a)$, the horizontal length scale imposed by the cylinder gap width; an Ekman number

$$E = \nu/f_0 H^2; \tag{20}$$

the Prandtl number

$$\sigma = \nu/\kappa; \tag{21}$$

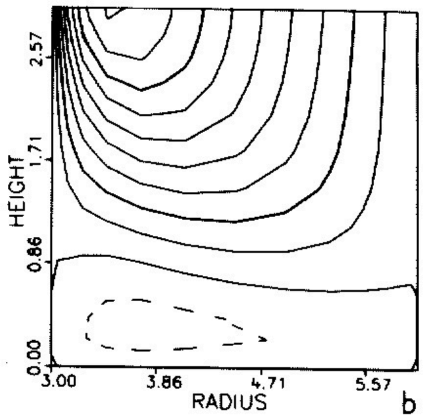
and the two aspect ratios

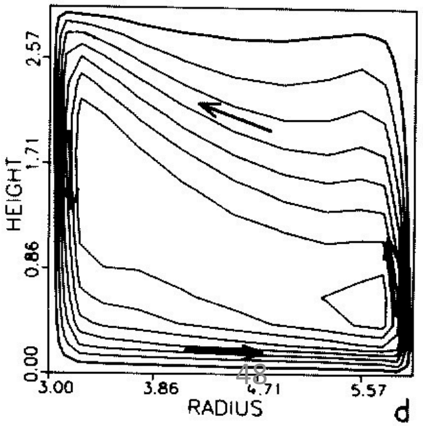
$$\varepsilon = H/L \tag{22}$$

$$\eta = L/\bar{r} \tag{23}$$

where $\bar{r}$ is a 'typical' radius, and η thus measures the effect of cylindrical curvature. It is also convenient to define a subsidiary dimensionless parameter—a 'curvature' Rossby number

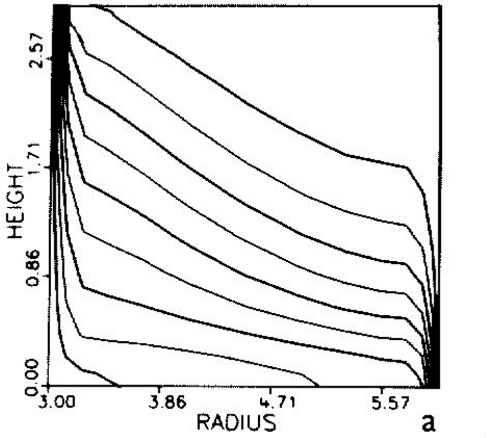
$$Ro_c = V/f_0\bar{r}, \;\sim S \tag{24}$$

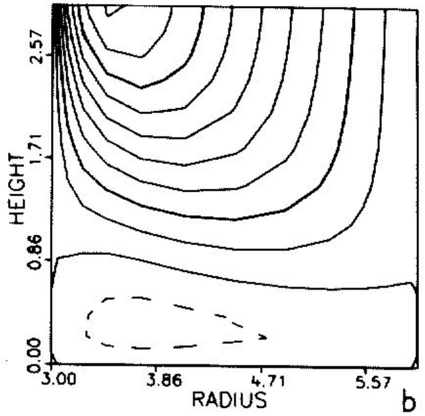


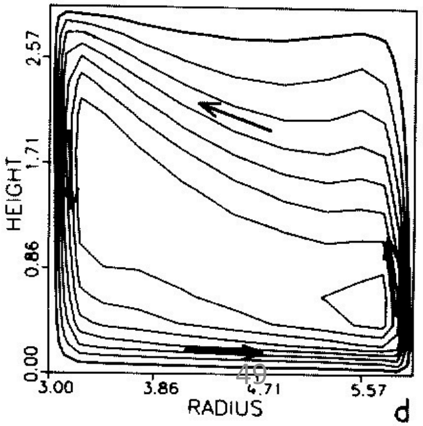


A scaling theory for super-rotation in a rotating annulus

[Read1986]







- Provided *Ra* and E^{-1} are >>1 then meridional flow is mostly confined to sidewall thermal and horizontal Ekman boundary layers with a distinct interior
  - Sidewall thermal boundary layer

$$\ell_T \sim L \left(\frac{Ra}{\varepsilon}\right)^{-1/4}$$

  - Ekman layer

$$\ell_E \sim H E^{1/2}$$

  - Define squared ratio
  - $Q = \left(\frac{\ell_T}{\ell_E}\right)^2 = Ra^{-1/2} E^{-1} \varepsilon^{-3/2} \propto \Omega$
- Properties of the circulation largely depend on the magnitude of *Q*

A scaling theory for super-rotation in a rotating annulus [Read1986]

- Up to ~6 different dynamical regimes can be identified
  (i) zero rotation ($Q = 0$);
  (ii) 'very weak' rotation ($0 \ll Q \ll \varepsilon^2\sigma^{-2}$);
  (iii) 'weak' rotation ($\varepsilon^2\sigma^{-2} \ll Q \ll 1$);
  (iv) 'moderate' rotation ($Q \sim 1$);
  (v) 'strong' rotation ($1 \ll Q \ll \varepsilon^{1/2}Ra^{1/6}$);
  (vi) 'very strong' rotation ($Q \gg \varepsilon^{1/2}Ra^{1/6}$).
- Weak/very weak rotation regime dominated by sidewall thermal layers so $\frac{\chi}{r} \sim \kappa\varepsilon^{3/4}Ra^{1/4}$

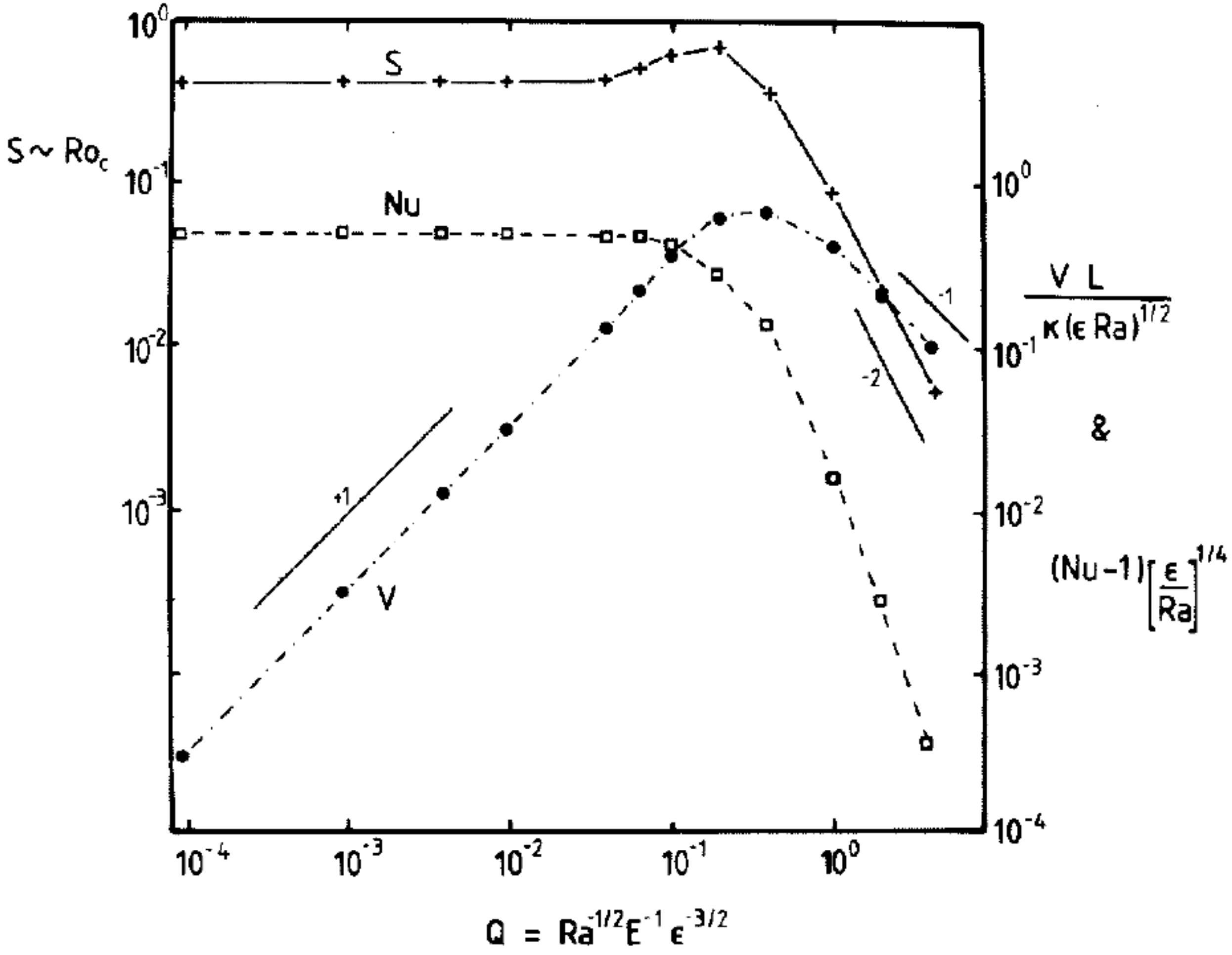
A scaling theory for super-rotation in a rotating annulus

- In regime (ii) get a nonlinear-Coriolis balance in zonal momentum equations which implies angular momentum is conserved

$$(1/r_*)J(\chi_*, m_*) = (\sigma^2 Q/\varepsilon^2)[\nabla\cdot\{r_*^2\nabla(m_*/r_*^2)\}]$$

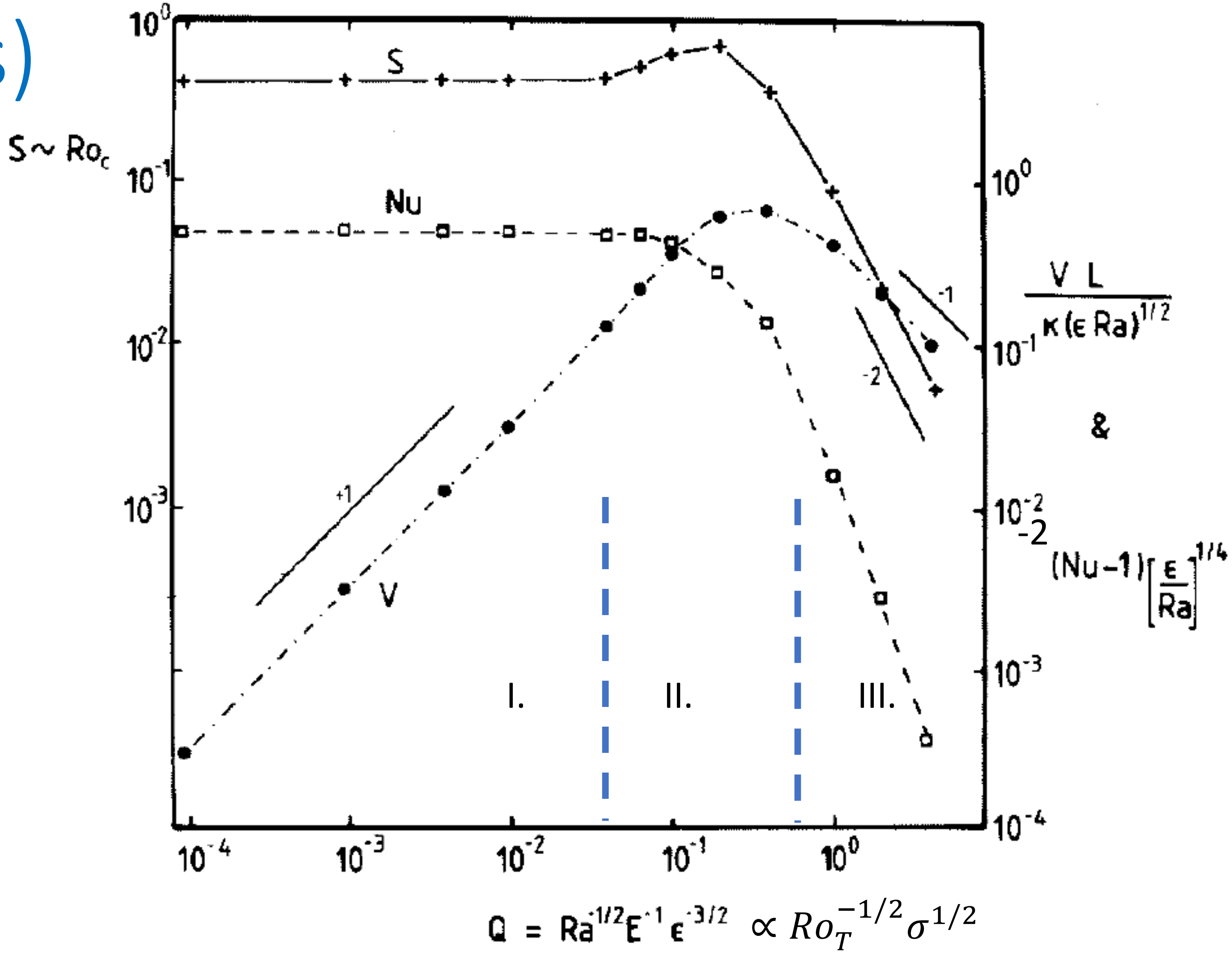
- Regime (iii) V scale depends on curvature η and ε but can be geostrophic or cyclostrophic

  ($Ro_c > 1$)



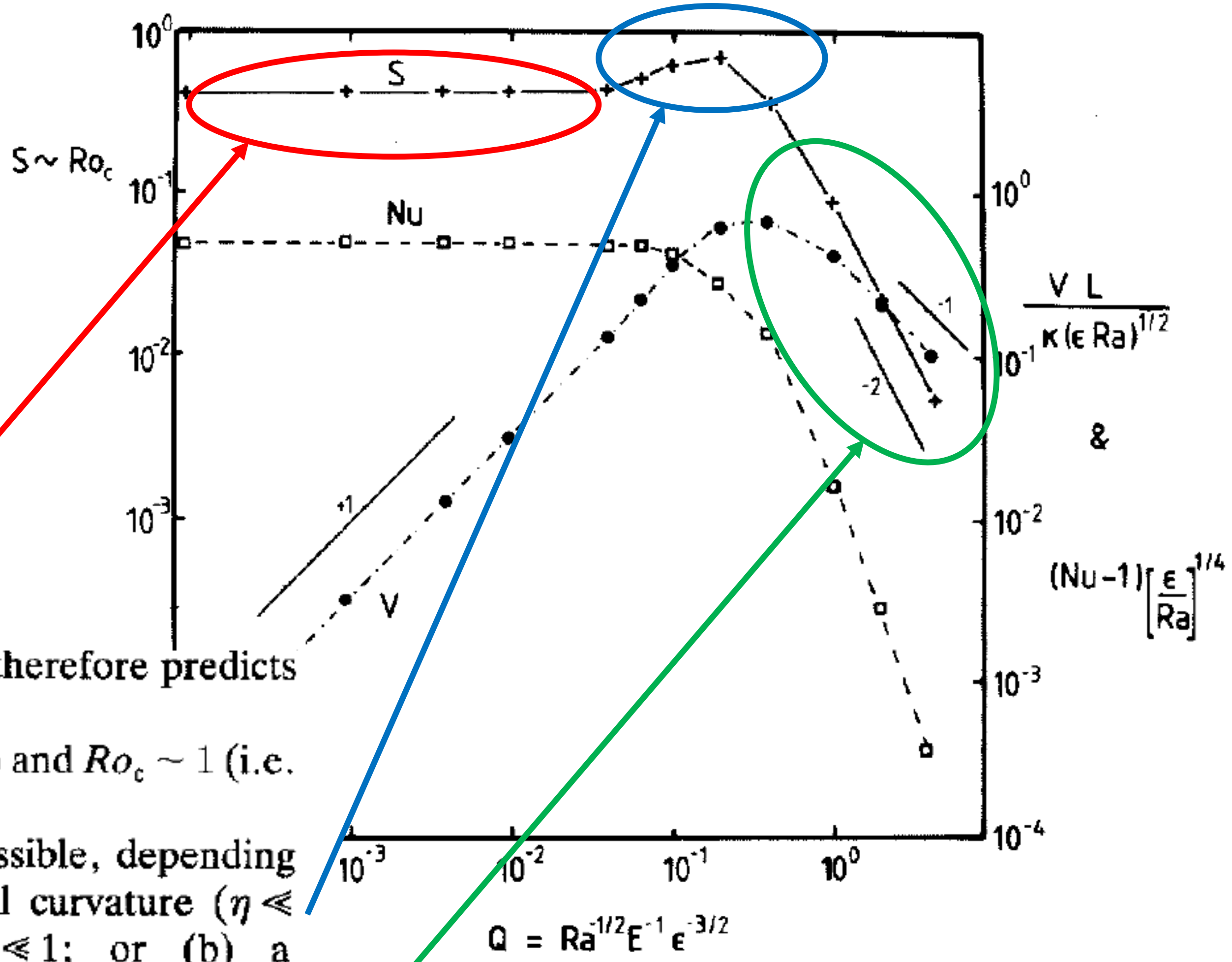
Trends in S (cylindrical annulus)

- 3 basic regimes
  - I. V. Slow rotation (~angular momentum conserving except in Ekman layer): S ~ constant
  - II. Moderate rotation (cyclostrophic/gradient wind and diffusive interior): S rises to shallow peak ($S \approx \varepsilon \eta^{1/2} \sigma^{-1/2} Q^{-1}$)
  - III. Rapid rotation (quasi-geostrophic): $S \sim Q^{-2} \sim \Omega^{-2}$



A scaling theory for super-rotation in a rotating annulus

- Circulation largely determined by interactions between boundary layers and interior flow
- Three main super-rotation regimes, depending upon *Q*



For the global super-rotation properties of the flow, the analysis therefore predicts three main regimes which may be summarized as follows:

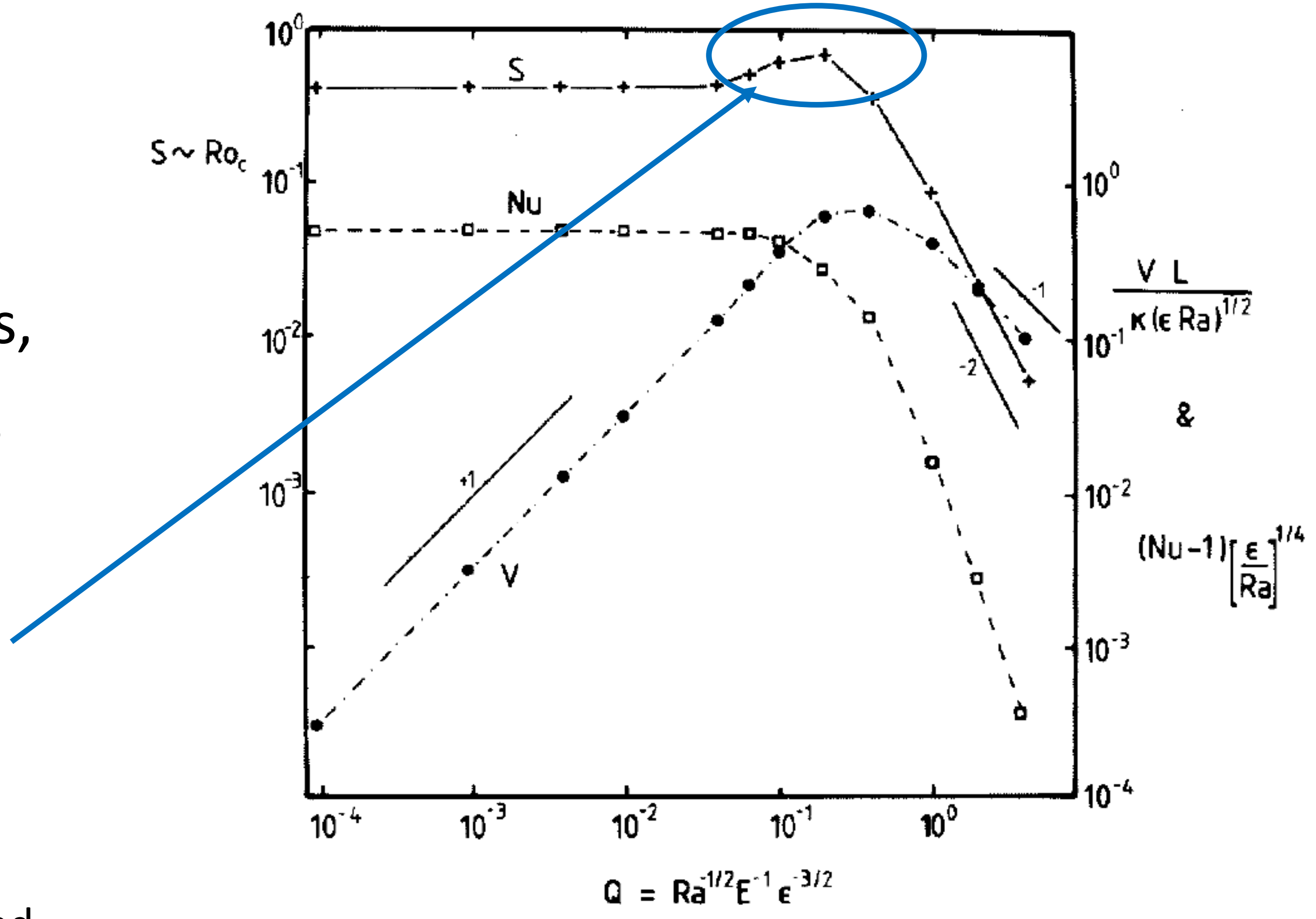
(I) 'slow' rotation ($Q \ll \varepsilon^2\sigma^{-2}$), where $V = f_0\bar{r}$ (i.e. proportional to Q) and $Ro_c \sim 1$ (i.e. the 'angular momentum conserving' regime);

(II) 'moderate' rotation ($\varepsilon^2\sigma^{-2} \ll Q < 1$), in which two cases are possible, depending upon ε, η and σ: (a) a 'weak' super-rotation regime for small curvature ($\eta \ll \sigma^2 Q\varepsilon^{-2}$), where $V = \kappa\varepsilon Ra^{1/2}Q^{1/2}L^{-1}$ and $Ro_c = \varepsilon\eta\sigma^{-1}Q^{-1/2} \ll 1$; or (b) a 'strong' super-rotation regime, obtained for large curvature ($\eta \gg \varepsilon^2\sigma^{-3}$) where $V = \nu(Ra\,\bar{r}H/\sigma)^{1/2}L^{-2}$ and $Ro_c = \varepsilon\eta^{1/2}\sigma^{-1/2}Q^{-1}$ (which can be $\gg 1$);

(III) 'rapid' rotation ($Q \gg 1$), where V is geostrophic and given by the thermal wind scale $V = \kappa Ra^{1/2}\varepsilon^{3/2}L^{-1}Q^{-1}$, so that $Ro_c = \varepsilon^2\eta\sigma^{-1}Q^{-2} \ll 1$.

A scaling theory for super-rotation in a rotating annulus

- Circulation largely determined by interactions between boundary layers and interior flow
- Three main super-rotation regimes, depending upon Q
  - Moderate rotation regime allows for two possibilities
    - Weak super-rotating regime has *u* in geostrophic balance
    - Strong super-rotating regime has *u* in cyclostrophic balance
      $$\frac{\partial(u^2)}{\partial z} \approx g\alpha \frac{\partial T}{\partial r}$$
    - Also requires $\sigma > \varepsilon^{2/3}\eta^{-1/3}$ so thermal diffusion is weak cf viscosity
    - $S \sim Ro_c = \varepsilon\eta^{1/2}\sigma^{-1/2}Q^{-1}$ so enhanced by large aspect ratio ε and curvature η



A numerical example

- Increase aspect ratio ε from 1 - 2
- Increase curvature η from 2/3 to 3/2
- S_{max} increases to 2.21 from 0.68
  - Close to predicted factor of $2\sqrt{2}$
- Peak angular velocity ω_{max} = 1.9 rad s^{-1}, ~10 x Ω!

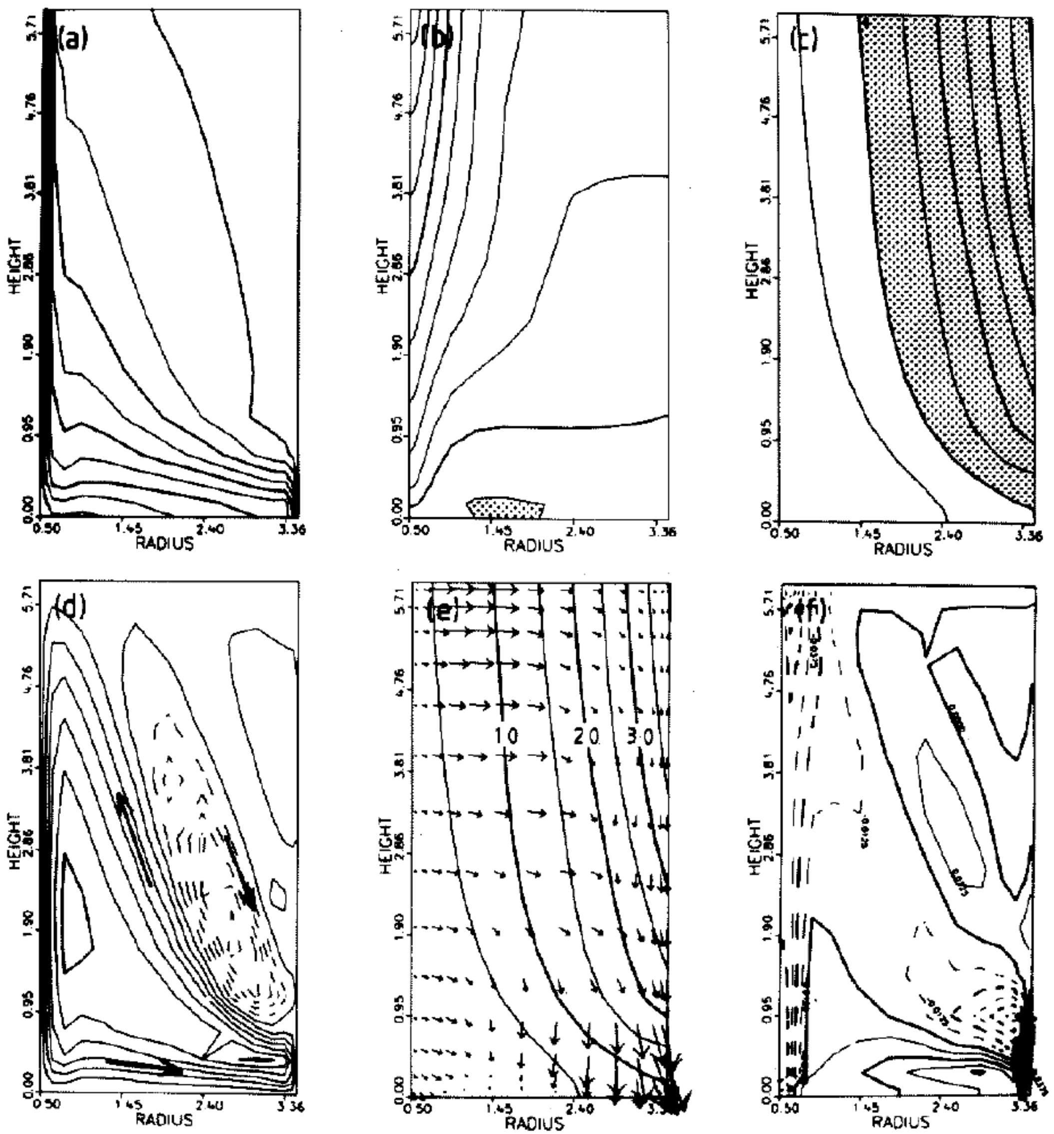
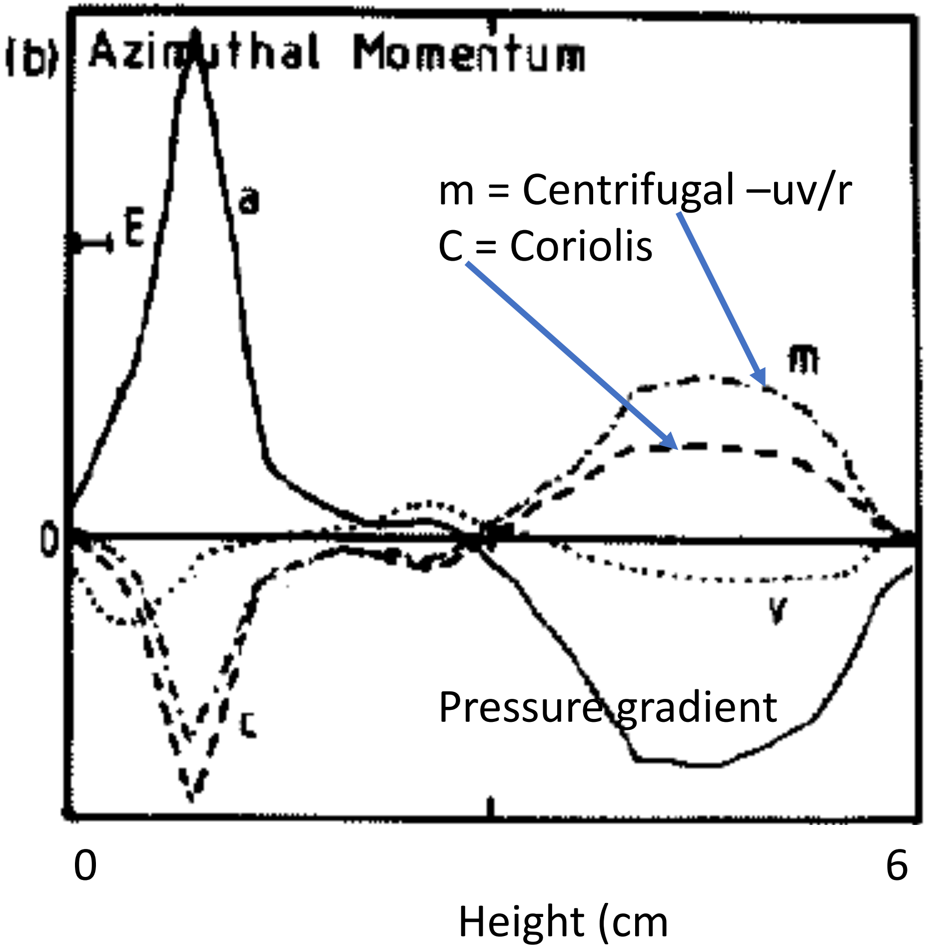


Figure 8. Contour maps of the steady-state fields for a numerical simulation of the thermally-driven axisymmetric circulation in a rotating fluid annulus with rigid, non-slip base, and stress-free side and top boundaries (case D, see text): (a) temperature (contour interval, 0·5 K); (b) γ (contour interval, 0·2 s^{-1}; the region where $\gamma < 0$ is shown shaded); (c) $m/\Omega b^2$ (contour interval, 0·5; the region where $m > \Omega b^2$ is shown shaded); (d) χ (contour interval, 0·01 $cm^3 s^{-1}$); (e) $m/\Omega b^2$ (see (c)) with vectors of **F** superimposed; (f) $-\nabla \cdot \mathbf{F}$ (contour interval, 0·0125 $cm^2 s^{-2}$). Negative contours are dashed.

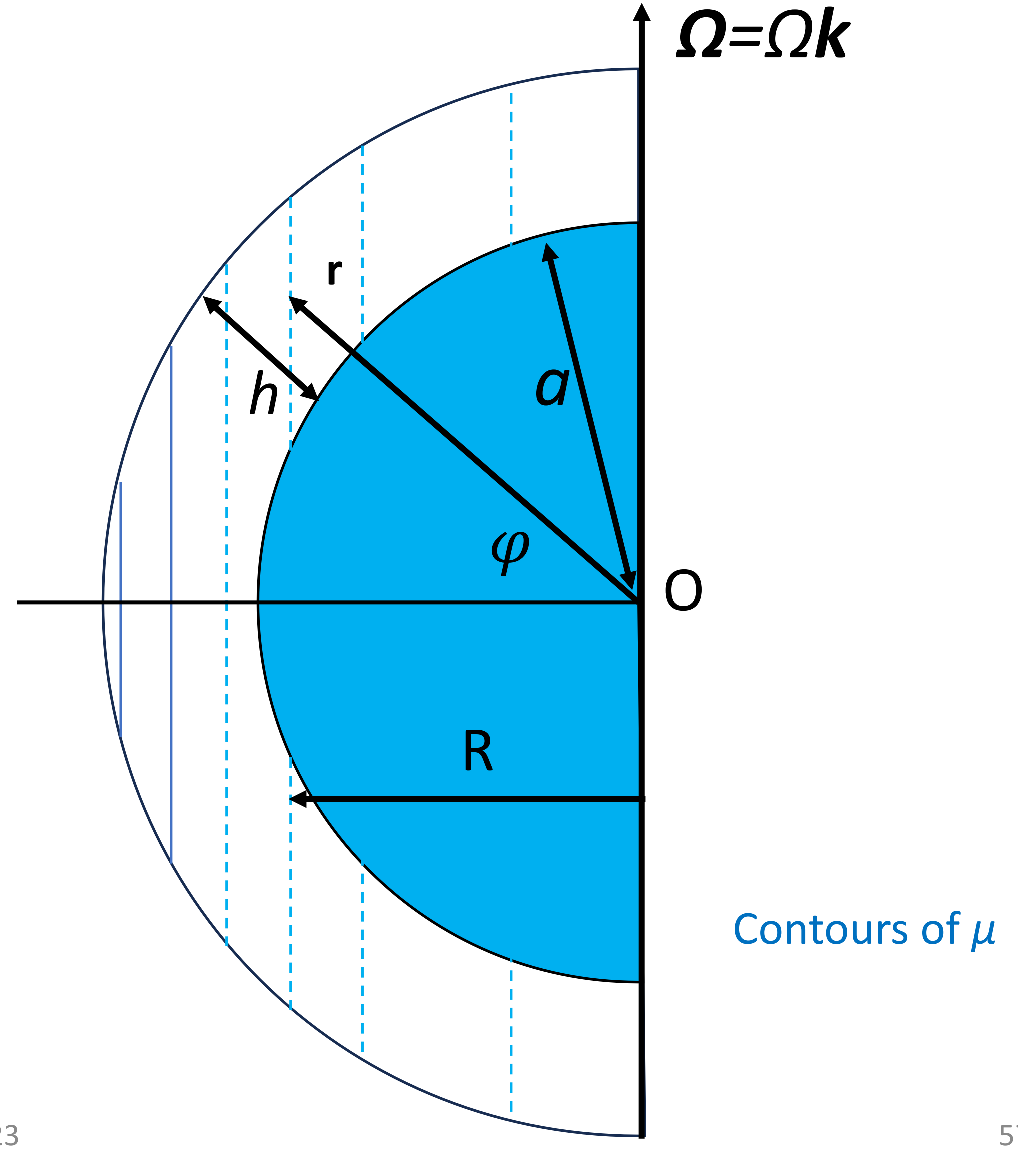
A numerical example

- Increase aspect ratio ε from 1 - 2
- Increase curvature η from 2/3 to 3/2
- S_{max} increases to 2.21 from 0.68
  - Close to predicted factor of $2\sqrt{2}$
- Peak angular velocity ω_{max} = 1.9 rad s^{-1}, ~10 x Ω!
- Centrifugal acceleration starts to dominate over Coriolis



Viscous AM fluxes in spherical geometry?

- For a DEEP spherical shell
$$m = (\Omega r\cos\varphi + u) r\cos\varphi$$
- the viscous flux term can be written as
$$\boldsymbol{F} = -\nu r^2 \cos^2\varphi \nabla\omega$$
$$= -\nu r^2\cos^2\varphi\nabla\left(\frac{m}{r^2\cos^2\varphi}\right)$$
- So
- $\boldsymbol{F}.\nabla m = -\nu\left[|\nabla m|^2 - \frac{1}{R}\frac{\partial}{\partial R}(m^2)\right]$
  - Where $R = r\cos\varphi$ is "cylindrical" radius
  - [Exercise: show this!]
- This means that ***F*** can be up-gradient for *m* in the *R* direction but not parallel to ***k*** (i.e. parallel to ***Ω***)



Viscous AM fluxes in spherical geometry?

- For a SHALLOW spherical shell, *h* << *a* and *r* ~ *a* so

$$m = (\Omega a \cos\varphi + u) a \cos\varphi$$

- The viscous flux term is written as

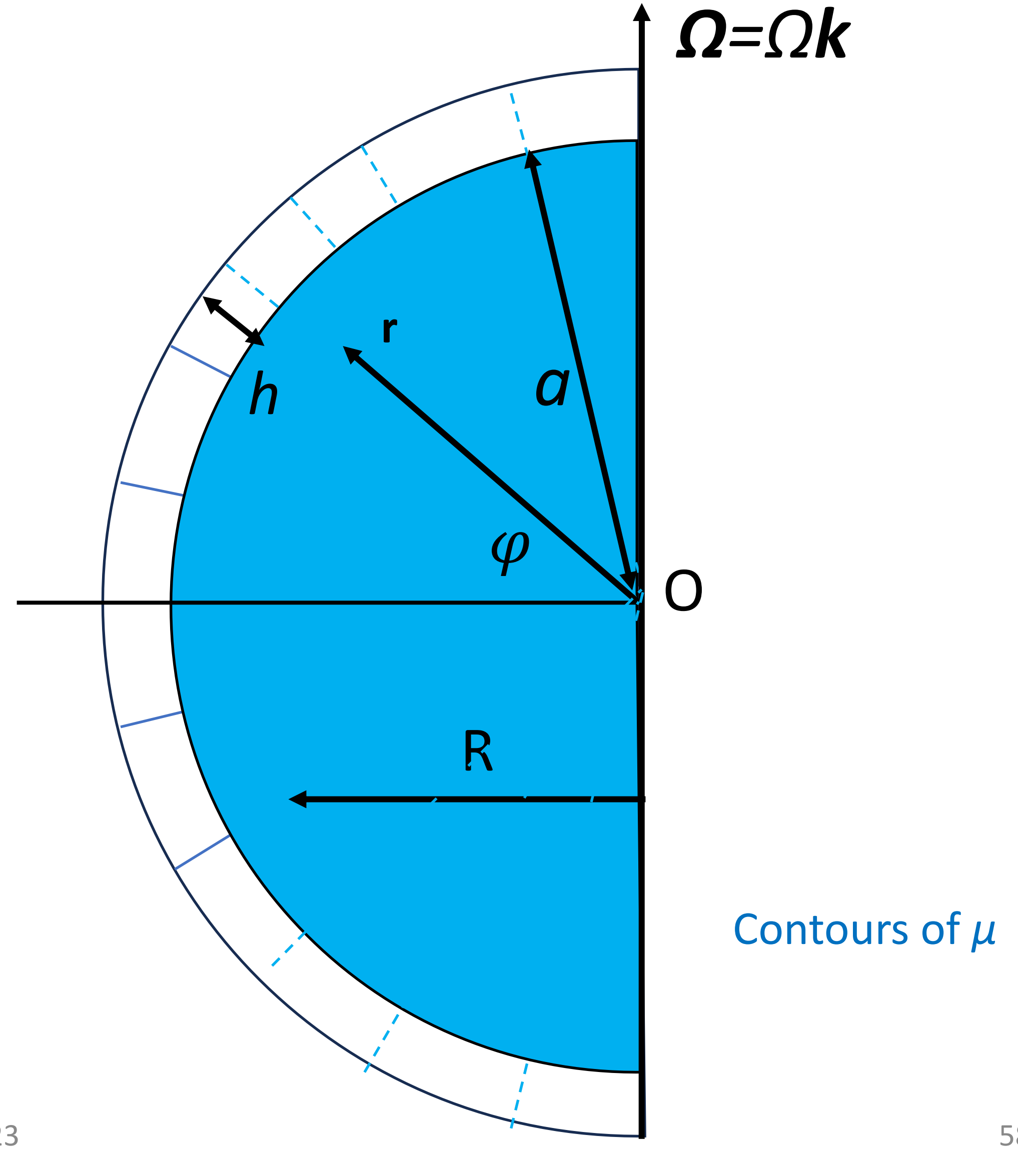
$$\boldsymbol{F} = -\nu a^2 \cos^2\varphi \nabla\omega$$
$$= -\nu a^2 \cos^2\varphi \nabla\left(\frac{m}{a^2\cos^2\varphi}\right)$$

  - Where $\omega = u/(a\cos\varphi)$ so that

$$\boldsymbol{F}.\nabla m = -\nu\left[|\nabla m|^2 - \frac{1}{a^2}\frac{\partial}{\partial\varphi}(m^2)\tan\varphi\right]$$

  - [Exercise: show this!]

- This means that ***F*** can be up-gradient for *m* in the φ direction but not in the local vertical
  - **NB Even at the equator!**



Summary I.

- Super-rotation is best defined with respect to angular momentum, not angular velocity
- Specific angular momentum is conserved by axisymmetric flows in the absence of viscosity and body forces
- Super-rotation requires non-axisymmetric eddies that transfer AM up-gradient
- Super-rotation is widely observed in various planetary circulations, but with widely differing magnitude
- Scaling arguments may be useful but probably depend on the system…?