Random stretching

December 7, 2000

1 Overview

In the previous lecture we emphasized that the destruction of tracer variance by molecular diffusivity relies on the increase of ∇c by stirring. Thus the term $\kappa\langle\nabla c'\cdot\nabla c'\rangle$ in the variance budget eventually becomes important, even though the molecular diffusivity κ is very small. One goal of this lecture is to understand in more detail how tracer gradients in a moving fluid are amplified by simple velocity fields. We will assume that $\kappa = 0$ so that there stirring without mixing. This is a good approximation provided that the smallest scale in the tracer field is much greater than the length $\ell = \sqrt{\kappa/\alpha}$ which we identified in lecture 1.

Gradient amplification is closely related to the stretching of material lines, a subject which was opened by Batchelor in 1952. A material line is a curve which consists always of the same fluid particles. Batchelor's main conclusion is that there is a timescale governing the ultimate growth of an infinitesimal line element, but no length scale other than that of the element itself. These dimensional considerations force the conclusion that the element grows exponentially,

$$\ell = \ell_0 e^{\gamma t}\,, \tag{1}$$

where γ is a constant.

Just as some close particle pairs separate exponentially, other pairs starting at distant points are brought close together. This might seem paradoxical until one recalls the folded tracer patterns evident in Welander's 1955 experiments (see the final figures in lecture 1). If two closely approaching particles are carrying different values of c then the gradient ∇c will be amplified.

Thus, as a corollary of (1) we expect that $|\nabla c| \sim |\nabla c_0| \exp(\gamma t)$. It is through this exponential amplification of the concentration gradients that the small molecular diffusivity κ is able eventually to destroy tracer variance.

The random Couette process

The simplest model of exponential stretching is the steady stagnation point flow, $\boldsymbol{u} = (\alpha x, -\alpha y)$. All line elements eventually grow exponentially in this simple flow. This example of exponential stretching gives the mistaken impression that hyperbolic stagnation points play an essential role in the process. To show that hyperbolic stagnation points are inessential, we consider stretching by the Couette flow $\boldsymbol{u} = (0, \beta y)$. If we release a material line element $\boldsymbol{\xi} = \ell_0(\cos\theta_1, \sin\theta_1)$ in this Couette flow then at time t the element is

$$\boldsymbol{\xi}(t) = \ell_0(\cos\theta_1 + \beta t \sin\theta_1, \sin\theta_1)\,. \tag{2}$$

The length of this element at t is

$$\ell^2(t) = \left[1 + \beta t \sin 2\theta_1 + \beta^2 t^2 \sin^2\theta_1\right] \ell_0^2\,. \tag{3}$$

Notice that when βt is large $\ell(t)$ grows linearly with time, which is very different from the exponential growth in (1).

However, suppose we stop the elongation in (2) at $t = \tau$ and renovate the process by starting a new Couette flow at a random angle to the first. We can implement this sudden change in direction by taking a new angle, say θ_2, in (2) and replacing ℓ_0 by $\ell_1 \equiv \ell(\tau)$. Thus the *random Couette process* is constructed by renovating at $t = n\tau$ with a fresh angle θ_n in each epoch. After n iterations

$$\ell^2(n\tau) = \prod_{k=1}^{n} s^2(\theta_n)\ell_0^2\,. \tag{4}$$

where the random stretching factor is $s^2(\theta) \equiv 1 + \beta\tau\sin 2\theta + \beta^2\tau^2\sin^2\theta$. In other words, the length of the element after at $t = n\tau$ is the product of n independent and uniformly distributed random stretches, $s(\theta_k)$ where θ_k is a random angle uniformly distributed in $[0, 2\pi]$.

Computing averages of the random product in (4) we discover that the asymptotic growth of the "average" length is exponential, as anticipated in

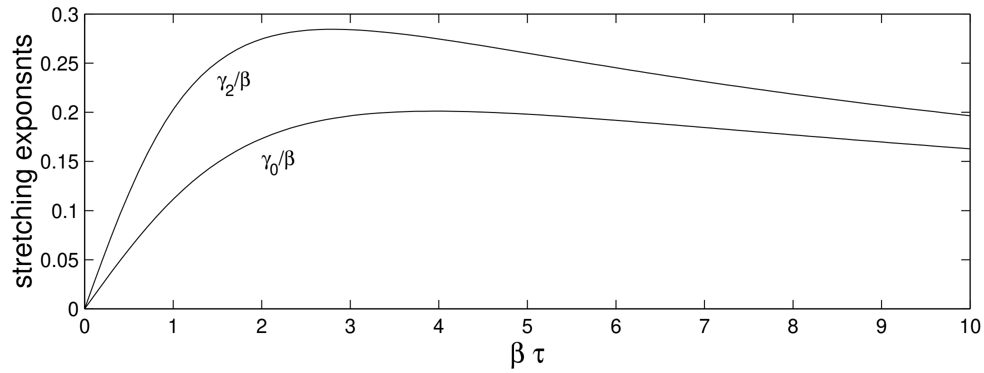

Figure 1: The exponents of the random Couette process, γ_0/β in problem 1.1 and γ_2/β in (7), as functions of $\beta\tau$.

(1). This exponential growth happens even though the elements grow only linearly with time in a steady Couette flow: the *random realignment* which happens at $t = n\tau$ is crucial in increasing the efficacy of stretching.

Since the average over θ of $s^2(\theta)$ is $\langle s^2 \rangle = 1 + (\beta^2\tau^2/2)$, the simplest characterization of stretching by the random Couette process is

$$\langle \ell^2(n\tau) \rangle = \left(1 + \frac{1}{2}\beta^2\tau^2\right)^n \ell_0^2 \,. \tag{5}$$

Noting that $n = t/\tau$, we emphasize the similarity to (1) by rewriting (5) as

$$\sqrt{\langle \ell^2(t) \rangle} = e^{\gamma_2 t}\ell_0 \,, \tag{6}$$

where the stretching exponent is

$$\gamma_2 = \frac{1}{2\tau}\ln\left(1 + \frac{\beta^2\tau^2}{2}\right) \,. \tag{7}$$

The exponent γ_2 in (7) has a nonmonotonic dependence on the nondimensional parameter $\beta\tau$: γ_2/β is maximized if $\beta\tau \approx 4$ (see figure 1). When the correlation time is small ($\beta\tau \ll 1$) we have $\gamma_2 \approx \beta^2\tau/4$; increasing the correlation time means more stretching because the velocity acts coherently for longer intervals. But in the other limit, $\beta\tau \to \infty$, we see that $\gamma_2 \to 0$. In this limit stretching is ineffective because advection by a persistent velocity means that the element spends a lot of time inefficiently aligned with the direction of the velocity.

Stretching exponents

Various measures of stretching are provided by the p'th-stretching exponent, γ_p. Following Drummond & Münch (1990), we define γ_p, as

$$\gamma_p \equiv \lim_{t\to\infty} \frac{1}{p\langle \ell^p \rangle} \frac{\mathrm{d}\langle \ell^p \rangle}{\mathrm{d}t}\,, \qquad p > 0\,. \tag{8}$$

Why should we care about these stretching exponents γ_p? Why not stop with γ_2, which is the easiest γ_p to evaluate? Why does the literature on random stretching emphasize γ_0, defined by

$$\gamma_0 \equiv \lim_{p\to 0} \gamma_p = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \langle \ln \ell \rangle\,, \tag{9}$$

so strongly? Answering these questions requires an excursion into the peculiar properties of multiplicative random variables (see section 2).

Problem 1.1. Show that for the two-dimensional random Couette process

$$\gamma_p = \frac{1}{p\tau} \ln \left[\frac{1}{2\pi} \oint \left(1 + \beta\tau \sin 2\theta + \beta^2\tau^2 \sin^2\theta\right)^{p/2} \mathrm{d}\theta \right], \tag{10}$$

and

$$\gamma_0 = \frac{1}{2\tau} \oint \ln\left[1 + \beta\tau \sin 2\theta + \beta^2\tau^2 \sin^2\theta\right] \mathrm{d}\theta\,, \tag{11}$$

$$= \frac{1}{2\tau} \ln\left(1 + \frac{\beta^2\tau^2}{4}\right)\,. \tag{12}$$

Compare the analytic results for γ_0 and γ_2 with a Monte Carlo simulation of random Couette line stretching (see figure 2).

Problem 1.2. Formulate and solve a renovation model based on randomly reorienting the straining flow $\psi = \alpha x y$ at $t = n\tau$. Calculate some stretching exponents. Are these exponents greater or less than α?

Problem 1.3. Generalize the Random Couette process to three dimensions. Show that

$$\gamma_2 = \frac{1}{2\tau} \ln\left[1 + \frac{\beta^2\tau^2}{3}\right] \text{.(check this!)} \tag{13}$$

2 Multiplicative random variables

We begin with some general remarks about multiplicative random processes, such as the random product in (4). Suppose that a random quantity, X, is

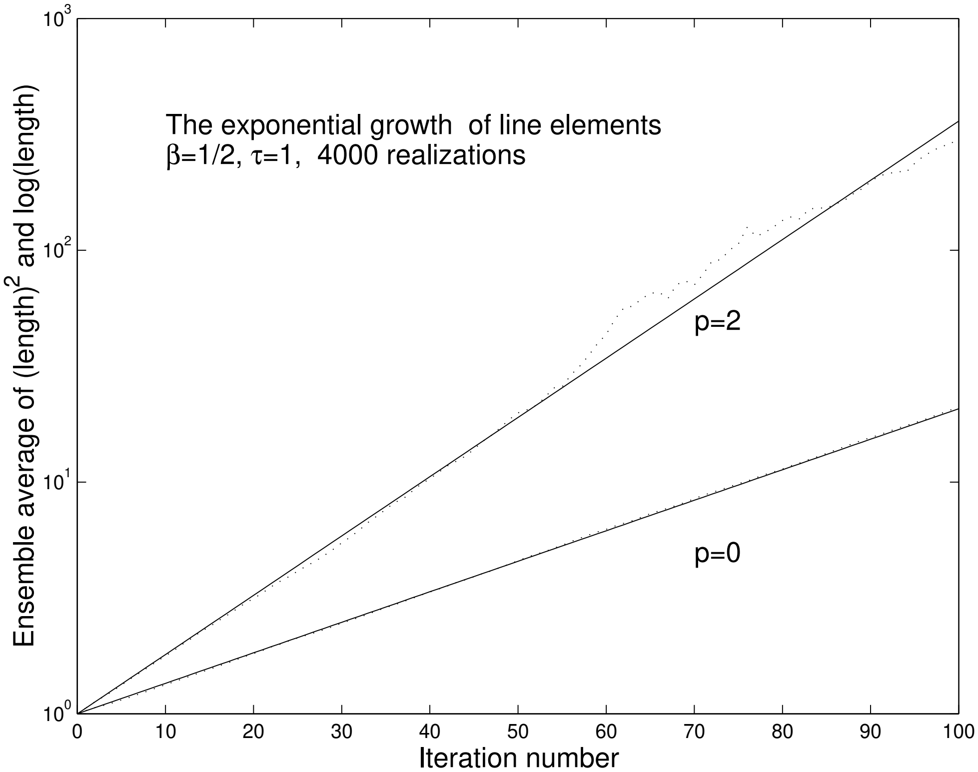

Figure 2: A comparison of the exponents γ_0 and γ_2 with a simulation (the dotted curves) of the random Couette process. To get reasonable agreement between the simulation and the analytic result in (7) one must ensemble average over a large number of realizations (4000 in the figure above). The discrepancies evident at large iteration number, $n = t/\tau$ can be reduced by using more realizations.

formed by taking the product of N independent and identically distributed random variables

$$X = x_1 x_2 \cdots x_N. \tag{14}$$

What can we say about the statistical properties of X?

The most nonintuitive aspect of X in (14) is the crucial distinction which must be made between the *mean* value of X and the *most probable* value of X. As an illustration, it is useful to consider an extreme case in which each x_k in (14) is either $x_k = 0$ or $x_k = 2$ with equal probability. Then the sample space consists of 2^N sequences of zeros and two's. For all but one those sequences, $X = 0$; in the remaining single case $X = 2^N$. Thus, the most probable (that is, most frequently occuring) value of X is

$$X_{\rm mp} = 0\,. \tag{15}$$

On the other hand, the mean of X is

$$\langle X \rangle \equiv \frac{\text{sum all the } X\text{'s from different realizations}}{\text{number of realizations}} = 1\,. \tag{16}$$

Notice that one can also calculate $\langle X \rangle$ by arguing that $\langle x_k \rangle = 1$ and, since the x_k's are independent, $\langle X \rangle = \langle x_k \rangle^N = 1$.

The example above is representative of multiplicative processes in that extreme events, although exponentially rare if $N \gg 1$, are exponentially different from typical or most probable events. Thus, for the *product* of N random variables the ratio $\langle X \rangle / X_{\rm mp}$ diverges exponentially as $N \to \infty$. On the other hand, for the *sum* of N random variables the most probable outcome is a good approximation of the mean outcome. Perhaps this is why people have an intuitive appreciation of sums, but find products confusing.

Now let us consider a more realistic example in which each x_k is either α or $1/\alpha$ with probability $1/2$. In this case

$$\langle x_k^p \rangle = \frac{\alpha^p + \alpha^{-p}}{2}\,, \tag{17}$$

and, since the x_k are independent, the p'th moment of X is

$$\langle X^p \rangle = \left(\frac{\alpha^p + \alpha^{-p}}{2} \right)^N . \tag{18}$$

We show in (21) that because $\langle \ln x_k \rangle = 0$ the most probable value of X is $X_{\rm mp} = 1$. For example, if $\alpha = 2$ then $\langle X \rangle = (5/4)^N$, while $X_{\rm mp} = 1$. Again, the most probable value differs exponentially from the mean value as $N \to \infty$.

The log-normal distribution

Because $X_{\rm mp}$ is so different from the $\langle X \rangle$ the problem of determining $\langle X \rangle$ via Monte Carlo simulation is difficult. For example, consider again the multiplicative process in which $x_k = 0$ or $x_k = 2$ with equal probability.

There are 2^n points in the sample space and with a monte Carlo calcualtion one would have to exhaust nearly all of the 2^N cases in order to obtain a reliable estimate of $\langle X \rangle = 1$. This exhaustion is necessary for the first example, in which $x_k = 0$ or 2. In the example of equation (17), provided that $\alpha \approx 1$, we can get a pretty good estimate of $\langle X \rangle$ with less than exhaustive enumeration of all sequences of the x_n's.

Begin by noting that

$$\ln X = \ln x_1 + \ln x_2 + \cdots + \ln x_N, \tag{19}$$

and so if $\ln x_k$ has finite variance then it follows from the Central Limit Theorem (CLT) that $\Lambda \equiv \ln X$ becomes normally distributed as $N \to \infty$.

The pitfall is in concluding that all the important statistical properties of Λ, and therefore of $X = \exp(\Lambda)$, can be calculated using the asymptotic log-normal distribution of X. This not the case because the PDF of Λ, $\mathcal{P}(\Lambda)$, is approximated by a Gaussian only in a central scaling region in which $|\Lambda| < cN^{1/2}$, where c is some constant which depends on the PDF of x_k. On the other hand, a reliable calculation of $\langle X^p \rangle = \langle \exp(p\Lambda) \rangle$ may require knowledge of the tail-structure of $\mathcal{P}(\Lambda)$.

To illustrate these difficulties, we use the example in which $\ln x_k = \pm \ln \alpha$ and $\langle \ln^2 x_k \rangle = \ln^2 \alpha$. Invoking the Central Limit Theorem, the asymptotic PDF of Λ is therefore

$$\mathcal{P}_{\mathrm{CLT}}(\Lambda) = \frac{1}{\sqrt{2\pi N \ln^2 \alpha}} \exp\left(-\Lambda^2 / 2N \ln^2 \alpha\right). \tag{20}$$

In the central scaling region, $\mathcal{P}(\Lambda) \approx \mathcal{P}_{\mathrm{CLT}}(\Lambda)$.

To determine X_{mp} we can consider $\Lambda = \ln X$, which is an additive process for which the mean and most probable coincide ($\langle \Lambda \rangle = \Lambda_{\mathrm{mp}}$) and consequently

$$\boxed{X_{\mathrm{mp}} = e^{\langle \ln X \rangle}.} \tag{21}$$

In our previous example with $\ln x_k = \pm \ln \alpha$, $\langle \ln X \rangle = 0$ and $X_{\mathrm{mp}} = 1$.

Continuing with this example, we now attempt to recover the exact result in (17) by substituting (20) into

$$\langle X^p \rangle \equiv \int_{-\infty}^{\infty} \mathrm{e}^{p\Lambda} \mathcal{P}\left(\Lambda\right) \, \mathrm{d}\Lambda. \tag{22}$$

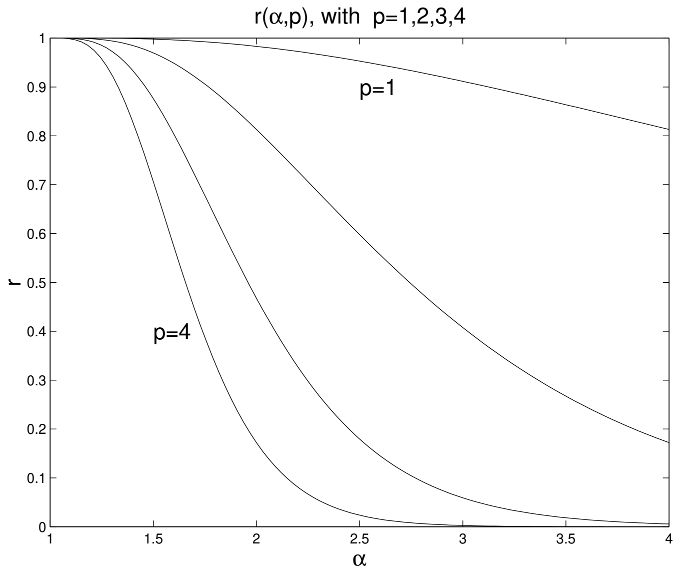

Figure 3: The function $r(\alpha, p)$ defined in (24). In order to accurately estimate $\langle X^p \rangle$ using the CLT one must have $r \approx 1$.

After the integration, one finds that

$$\langle X^p \rangle_{\mathrm{CLT}} = \exp\left(Np^2 \ln^2 \alpha/2\right) . \tag{23}$$

To assess the error we form the ratio of the exact result to the approximation:

$$\langle X^p \rangle / \langle X^p \rangle_{\mathrm{CLT}} = r^N, \quad \text{where} \quad r \equiv \frac{1}{2} \exp\left(-p^2 \ln^2 \alpha/2\right) \left(\alpha^p + \alpha^{-p}\right) . \tag{24}$$

When $r(\alpha, p)$ is close to 1, the error is tolerable in the sense that $\ln\langle X^p \rangle_{\mathrm{CLT}}$ is close to $\ln\langle X^p \rangle$; the function $r(\alpha, p)$ is shown in figure 3.

For example, with $\alpha = 2$, the exact result is $\langle X \rangle = (5/4)^N$ while $\langle X \rangle_{\mathrm{CLT}} = (1.27)^N$. However the second moment $p = 2$, is seriously in error. As a general rule, $\langle X^p \rangle_{\mathrm{CLT}}$ is a reliable estimate of $\langle X^p \rangle$ provided that $p^2 \langle \ln^2 x_k \rangle < c$, where c is the constant which determines the width of central scaling region, $|\Lambda| < cN^{1/2}$, in which $\mathcal{P}(\Lambda) \approx \mathcal{P}_{\mathrm{CLT}}(\Lambda)$. We conclude that the complete analysis of a random multiplicative quantity cannot be reduced to the Central Limit Theorem merely by taking a logarithm.

Stretching exponents again: why is γ_0 important?

Equation (21) is a very important result for multiplicative random variables: *to obtain the most probable value of* X*, exponentiate* $\langle \ln X \rangle$. This explains why there is so much attention paid to $\langle \ln[\ell(t)/\ell_0] \rangle$. The average of the logarithm enables one to estimate the stretching of a *typical* line element. Of course, the typical line element may not make a large contribution to the dissipation $\kappa \langle \nabla c' \cdot \nabla c' \rangle$. Thus our earlier focus on ℓ^2 in (5) and (7) was not wasted, but it was not complete either.

3 Material line elements and tracer gradients

Now we return to fluid mechanics and discuss random stretching more systematically. Using a geometric argument, see figure 4, we can give a proof-by-intimidation that a material line element, $\boldsymbol{\xi}(\boldsymbol{x}, t)$, attached to a fluid element evolves according to

$$\frac{D\boldsymbol{\xi}}{Dt} = (\boldsymbol{\xi} \boldsymbol{\cdot} \boldsymbol{\nabla}) \boldsymbol{u} \,. \tag{25}$$

The field of line elements can be visualized a collection of tiny straight arrows attached to each moving particle of fluid. Then (25) describes the evolution of this collection of arrows. Notice that (25) refers to an *infinitesimal* line element $\boldsymbol{\xi}$. If the length of a material line is comparable to the scale of $\boldsymbol{u}$ there is no longer a simple relation between the stretching of the material line and *local* properties of $\boldsymbol{u}$, such as $\boldsymbol{\nabla u}$.

Taking the gradient of the tracer equation

$$\frac{Dc}{Dt} = 0 \,, \tag{26}$$

gives

$$\frac{D\nabla c}{Dt} = -(\nabla c \cdot \nabla) \boldsymbol{u} \,. \tag{27}$$

Despite the difference in the sign of the right hand sides of (25) and (27) there is a close connection between the solutions of the two equations.

To emphasize the connection between $\boldsymbol{\nabla} c$ and $\boldsymbol{\xi}$, we mention the conservation law

$$\frac{D}{Dt} (\nabla c \cdot \boldsymbol{\xi}) = 0 \,. \tag{28}$$

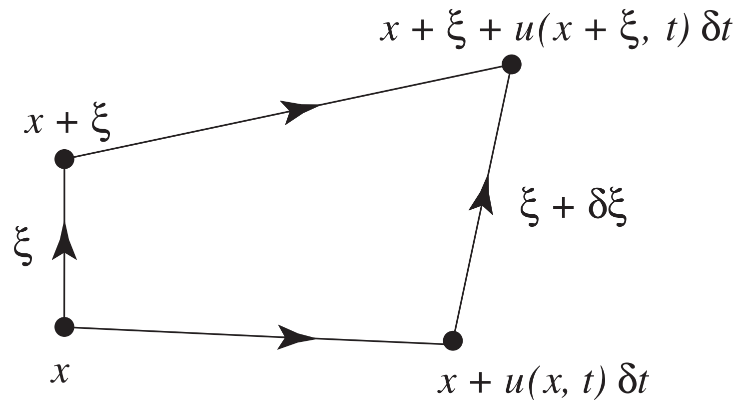

Figure 4: The line element $\boldsymbol{\xi}$ is short enough to remain straight and to experience a strain which is uniform over its length during the time δt. Proof by intimidation of (25) : $\delta\boldsymbol{\xi} = [u(\boldsymbol{x}+\boldsymbol{\xi},t) - u(\boldsymbol{x},t)]\delta t$, and take $(\delta t, \boldsymbol{\xi}) \to 0$.

(Meteorologists and oceanographers might recognize (28) as a relative of potential vorticity conservation.) In section 5 we use (28) is used to deduce $\boldsymbol{\nabla} c$ from $\boldsymbol{\xi}$.

The easy way to prove (28) is to consider a pair of particles separated by a small displacement $\boldsymbol{\xi}$. If the concentration carried by the first particle is c_1, and that of the second particle is $c_2 = c_1 + \mathrm{d}c$, then $\mathrm{d}c = \boldsymbol{\xi}\cdot\nabla c$. Thus (28) is equivalent to the "obvious" fact that dc is conserved as the two particles move.

The difficult way to prove (28) is to take the dot product of $\boldsymbol{\nabla} c$ with (25) and add this to the dot product of $\boldsymbol{\xi}$ with (27). Performing some nonobvious algebra, perhaps with Mathematica or Maple, one can eventually simplify the mess to (28). Suffering through this tedious exercise will convince the student that the earlier, easy proof is worthy of serious attention.

Eulerian versus Lagrangian: the golden rule

Particle trajectories, $\boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{x}_0)$, are determined by solving the differential equations

$$\frac{D\boldsymbol{x}}{Dt} = \boldsymbol{u}(\boldsymbol{x}, t)\,, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0\,. \tag{29}$$

The solution of the differential equation above defines the particle position, $\boldsymbol{x}$, as a function of the two independent variables, $\boldsymbol{x}_0$ and t. Using this time-dependent mapping between $\boldsymbol{x}$ and $\boldsymbol{x}_0$, we can take a problem posed in terms of $\boldsymbol{x}$ and t (the Eulerian formulation) and change variables to obtain an equivalent formulation in terms of $\boldsymbol{x}_0$ and t (the Lagrangian formulation).

In the Eulerian view, the independent variables are $\boldsymbol{x} = (x, y, z)$ and t. The convective derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\,, \tag{30}$$

is a differential operator involving all of the independent variables.

In the Lagrangian view, the independent variables are $\boldsymbol{x}_0$ and t' and $\boldsymbol{x}(\boldsymbol{x}_0, t')$ is a dependent variable. As an accounting device, the time variable is decorated with a prime to emphasize that a t'-derivative is means that the independent variables are $\boldsymbol{x}_0$. To move between the Eulerian and Lagrangian representations notice that

$$\frac{\partial t}{\partial t'} = 1\,, \quad \text{and} \quad \frac{\partial}{\partial t'}(x, y, z) = (u, v, w)\,. \tag{31}$$

The second equation above is the definition of velocity, $\boldsymbol{u} = (u, v, w)$.

Using (31), the rule for converting partial derivatives is

$$\boxed{\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t'}\frac{\partial}{\partial z} = \frac{D}{Dt}\,.} \tag{32}$$

Equation (32) is the golden rule which enables us to interpret expressions such as

$$\frac{D}{Dt}\text{unknown} = \text{RHS}\,, \tag{33}$$

in either Eulerian or Lagrangian terms. Using the golden rule we can dispense with the prime which decorates the Lagrangian time variable: we just remember that D/Dt is freighted with both a Lagrangian and an Eulerian interpretation.

In the Eulerian interpretation we must express the RHS in (33) as a function of x, y, z and t and use the Eulerian definition of the convective derivative in (30). Then (33) is a *partial* differential equation for the unknown.

In the Lagrangian interpretation D/Dt is the same as a simple time derivative and we must express the RHS of (33) as a function of x_0, y_0, z_0 and t. Then (33) is a *ordinary* differential equation for the unknown.

Motion is equivalent to mapping

We obtained (25) using the geometric argument in figure 4. Now we admire some different scenery by taking an algebraic path to (25). Our itinerary emphasizes that the solutions of (29) define a mapping of the space $\boldsymbol{x}_0$ of initial coordinates onto the space $\boldsymbol{x}$, and hence the title of this section.

Using indicial notation (summation implied over repeated indices), it follows from the chain rule that

$$\mathrm{d}\boldsymbol{x}_i = \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{x}_{0j}} \mathrm{d}\boldsymbol{x}_{0j}. \tag{34}$$

Taking the time derivative of (34), and keeping in mind that $\boldsymbol{x}_{0j}$ is independent of t, gives

$$\frac{D}{Dt}(\mathrm{d}\boldsymbol{x}_i) = \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_{0j}} \mathrm{d}\boldsymbol{x}_{0j} = \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_{0j}} \frac{\partial \boldsymbol{x}_{0j}}{\partial \boldsymbol{x}_k} \mathrm{d}\boldsymbol{x}_k = \frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j} \mathrm{d}\boldsymbol{x}_j\,. \tag{35}$$

(We have used the golden rule.) Making the identification $\mathrm{d}\boldsymbol{x} \to \boldsymbol{\xi}$ we obtain (25).

The motion of a fluid defines a family of mappings from the space of initial coordinates, $\boldsymbol{x}_0$, onto the space of coordinates $\boldsymbol{x}$. At $t = 0$ this is just the identity map but as t increases the map from $\boldsymbol{x}_0$ to $\boldsymbol{x}$ can become very complicated. Equation (34) defines the Jacobian matrix,

$$\boldsymbol{\mathcal{J}}_{ij} \equiv \frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{x}_{0j}}\,, \tag{36}$$

of the map.

With these algebraic formalities we have given an alternative derivation of (25) and, as a bonus, we have also found a representation of the solution:

$$\boldsymbol{\xi} = \boldsymbol{\mathcal{J}} \boldsymbol{\xi}_0\,. \tag{37}$$

The expression above is *Cauchy's* solution of (25).

In (37) there is no assumption that the flow is incompressible. If the flow is incompressible (i.e., if $\nabla \cdot \boldsymbol{u} = 0$) then mapping from $\boldsymbol{x}_0$ to $\boldsymbol{x}$ conserves volume. In this case, $\det \boldsymbol{\mathcal{J}} = 1$.

Problem 3.1. Solve the line-stretching equation (25) in the special case where $\boldsymbol{u}$ is a steady unidirectional two-dimensional velocity field, $\boldsymbol{u} = [u(y), 0]$.

Solution. Begin by noticing that the solution of (29) is

$$x = x_0 + u(y)t\,, \qquad y = y_0\,. \tag{38}$$

Thus it is a simple matter to express (x, y) in terms of (x_0, y_0) and vice versa.

The line-stretching equation, (25), has the same form as (33). Using components, $\boldsymbol{\xi} = (\xi, \eta)$, we have

$$\frac{D\xi}{Dt} = \eta u'(y_0)\,, \qquad \frac{D\eta}{Dt} = 0\,. \tag{39}$$

Using the golden rule we view (39) in Lagrangian varaibles so that we have an ordinary differential equation with the solution

$$\xi = \xi_0(x_0, y_0) + t\eta_0(x_0, y_0)u'(y_0)\,, \qquad \eta = \eta(x_0, y_0)\,. \tag{40}$$

Using (38), we can write (40) in terms of Eulerian variables as

$$\xi = \xi_0[x - u(y)t, y] + t\eta_0[x - u(y)t, y]u'(y)\,, \quad \eta = \eta_0[x - u(y)t, y]\,. \tag{41}$$

We can alternatively view (39) in terms of Eulerian variables and in this case we are confronted with the partial differential equations

$$\frac{\partial\xi}{\partial t} + u(y)\frac{\partial\xi}{\partial x} = \eta u'(y)\,, \qquad \frac{\partial\eta}{\partial t} + u(y)\frac{\partial\eta}{\partial x} = 0\,. \tag{42}$$

It is easy to check by substitution that (41) is the solution of (42).

Problem 3.2. Consider a one-dimensional compressible velocity $u = \sin x$. Solve the line-stretching equation

$$\xi_t + \sin x\, \xi_x = \xi \cos x\,, \quad \xi(x, 0) = 1\,, \tag{43}$$

with the initial condition that $\xi(x, 0) = 1$.

Solution. Begin by observing that the density $\rho(x, t)$ satisfies

$$\rho_t + (\sin x\, \rho)_x = 0 \qquad \rho(x, 0) = 1. \tag{44}$$

It is easy to show by substitution that the solutions of (43) and (44) are related $\rho(x, t) = 1/\xi(x, t)$. The physical interpretation of this result should be obvious...

To solve (43), we follow the route outlined in section 3 by determining the mapping from the initial space, x_0, to the space $x(x_0, t)$. This means we solve

$$\frac{Dx}{Dt} = \sin x, \qquad x(0, x_0) = x_0\,. \tag{45}$$

Using separation of variables we find that

$$\tan(x/2) = e^t \tan(x_0/2)\,, \tag{46}$$

which enables us to determine x given x_0, or vice versa. Figure 5 shows how the mapping from x_0 to x evolves as t increases. The Jacobian of the mapping in (46) is

$$\frac{\mathrm{d}x}{\mathrm{d}x_0} = \frac{1}{\cosh t - \cos x_0 \sinh t} = \cosh t + \cos x \sinh t\,. \tag{47}$$

It is easy to check that $\xi = \mathrm{d}x/\mathrm{d}x_0$ is the solution of (43).

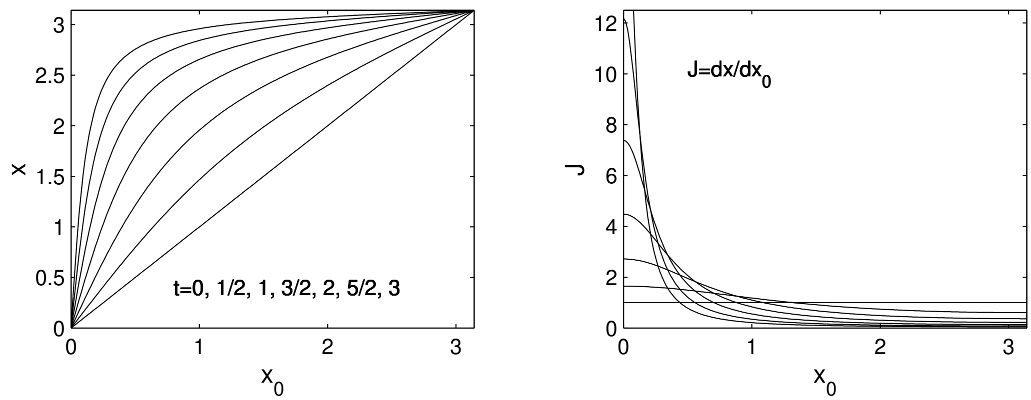

Figure 5: The left panel shows the mapping from x_0 to x at the indicated times. The interval $0 < x_0 < \pi$ is compressed into the neighbourhood of $x = \pi$. The right panel shows $J(x_0, t)$ at the same times. Notice that an element which starts at say, $x_0 = 1/2$, is first stretched ($J > 1$) but then ultimately compressed ($J < 1$) as the particle approaches $x = \pi$.

Problem 3.3. Consider one-dimensional line-element stretching produced by an ensemble of renovating sinusoidal velocity fields,

$$u = \sin(x + \varphi_n) \qquad \text{if} \qquad (n-1)\tau < t < n\tau\,. \tag{48}$$

The random phase, $0 < \varphi_n < 2\pi$, is reset at $t = n\tau$.

Solution. We follow the stretching of a line element attached to a particle which moves in a particular realization of this velocity field. We denote location of this particle at $t = n\tau$ by a_n, and the length of the attached line element at this time by ℓ_n. Then the stretching of the line element is given by the random product

$$\ell_n = J(a_{n-1})J(a_{n-2})\cdots J(a_0)\ell_0\,, \tag{49}$$

where the Jacobian is

$$J(a) \equiv \frac{1}{\cosh\tau - \cos a \sinh\tau}\,. \tag{50}$$

Because the phase is reset at $t = n\tau$, each $J(a_n)$ in (49) is independent of the others. Moreover, because of spatial homogeneity, each a_n is uniformly distributed with $0 < a_n < 2\pi$.

Equation (49) expresses the length of a material line element at $t = n\tau$ as a product of n random numbers. Following our discussion of multiplicative random variables, we first calculate γ_0 by taking the logarithm of (49):

$$\ln(\ell_n/\ell_0) = \sum_{k=0}^{n-1} \ln J(a_k)\,, \tag{51}$$

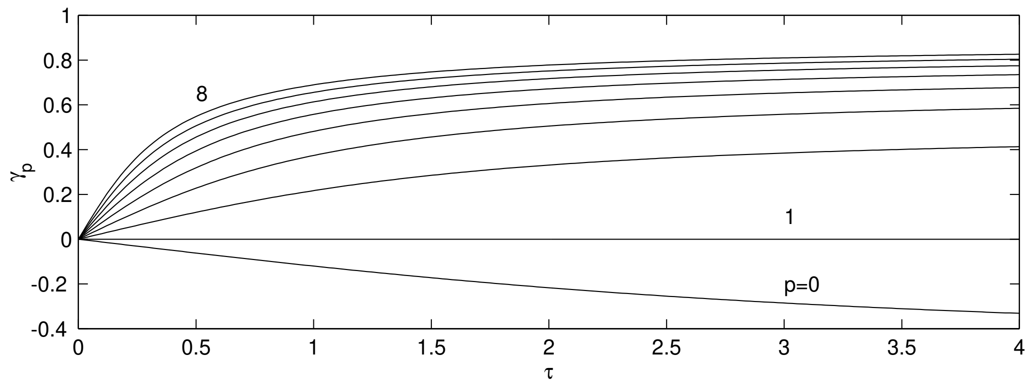

Figure 6: The stretching exponents $\gamma_p(\tau)$, with $p = 0,\ 1, \cdots,\ 8$ calculated using (58) .

Thus, the mean of $\ln(\ell_n/\ell_0)$ is

$$\langle \ln(\ell_n/\ell_0)\rangle = n\langle \ln J\rangle \,, \tag{52}$$

where

$$\langle \ln J\rangle = \oint \ln\left[J(a)\right]\, \frac{\mathrm{d}a}{2\pi} = -\ln\left[\cosh(\tau/2)\right] \,. \tag{53}$$

Because $\langle(\ln J)^2\rangle$ is finite, the central limit theorem applies and we conclude that as $n \to \infty$, $\ln(\ell_n/\ell_0)$ is approximately normally distributed with the mean value $n\langle \ln J\rangle$.

Moreover, we can conclude from the central limit theorem that the *most probable* value of ℓ_n/ℓ_0 is

$$(\ell_n/\ell_0)_{\mathrm{mp}} \approx e^{\langle \ln(\ell_n/\ell_0)\rangle} = e^{\gamma_0 t} \,, \tag{54}$$

where, since $n = t/\tau$,

$$\gamma_0 = -\ln[\cosh(\tau/2)]/\tau < 0 \,. \tag{55}$$

The result in (54) is remarkable because it implies that most of the line elements in this compressible flow exponentially contract (rather than stretch) as $t \to \infty$!

Exponential contraction of most material lines is incomplete disagreement with the spirit of Batchelor's result in (1), where $\gamma > 0$. The result above, that $\gamma_0 < 0$, is a special consequence of the *compressible* velocity field used in (48). (For a discussion of compressible velocities in a space of arbitrary dimension, see Chertkov et al. (1998).) This example shows that one cannot take exponential stretching for granted.

How is contraction in the length of most material elements compatible with conservation of the total length of the x-axis? Even though most elements become exponentially small as $t \to \infty$, a few elements become exponentially large. Thus most of the length accumulates in exponentially rare, but exponentially long, line elements. This is an elementary example of an *inverse cascade* i.e., the spontaneous appearence of large-scale structures (big line elements). To demonstrate length conservation, we can compute the mean (as opposed the most probable) length of an element. The mean length is

$$\langle \ell_n \rangle = \langle J \rangle^n \ell_0 \,, \tag{56}$$

where $J(a)$ is defined in (50) and

$$\langle J \rangle = \oint J(a) \frac{\mathrm{d}a}{2\pi} = 1 \,. \tag{57}$$

Thus, the mean length of an element is constant, even though most elements exponentially contract.

One can show further that for integer values of p the stretching exponents of this one-dimensional model are given by

$$\gamma_p = \ln \left[P_{p-1}(\cosh \tau) \right] / p\tau \,, \tag{58}$$

where P_m is the m'th Legendre polynomial (see figure 6).

4 Two-dimensional incompressible flow

In the case of a two-dimensional incompressible flow there is a streamfunction $\psi = \psi(\boldsymbol{x}, t)$ such that $\boldsymbol{u} = (u, v) = (-\psi_y, \psi_x)$. In terms of ψ, (25) can be written as:

$$\frac{D\boldsymbol{\xi}}{Dt} = \boldsymbol{W}\boldsymbol{\xi}, \qquad \text{where} \qquad \boldsymbol{W} \equiv \begin{pmatrix} -\psi_{xy} & -\psi_{yy} \\ \psi_{xx} & \psi_{xy} \end{pmatrix}. \tag{59}$$

The trace of $\boldsymbol{W}$ is zero and the determinant is $\det(\boldsymbol{W}) = \psi_{xx}\psi_{yy} - \psi_{xy}^2$. The solution of (59) can be written as

$$\boldsymbol{\xi} = \exp\left(\int_0^t \boldsymbol{W}(t')\,\mathrm{d}t' \right) \boldsymbol{\xi}_0 \,. \tag{60}$$

Thus, using (37), we obtain a fundamental connection between $\boldsymbol{\mathcal{J}}(t)$ and $\boldsymbol{W}(t)$:

$$\boldsymbol{\mathcal{J}}(t) = \exp\left(\int_0^t \boldsymbol{W}(t')\,\mathrm{d}t' \right). \tag{61}$$

Because $\operatorname{tr} \boldsymbol{W} = 0$ it follows[1] that $\det \boldsymbol{\mathcal{J}} = 1$. This is, of course, just another way of saying that if the flow is incompressible then the map from $\boldsymbol{x}_0$ to $\boldsymbol{x}$ is area preserving.

The steady case

Because (59) is linear the solution is straightforward if the velocity field in the Lagrangian frame is *steady*. Thus

$$\boldsymbol{\xi}(t) = \mathrm{e}^{\gamma t}\hat{\boldsymbol{\xi}}, \qquad \Rightarrow \qquad \gamma = \pm\sqrt{-\det \boldsymbol{W}}\,, \tag{62}$$

where

$$\det \boldsymbol{W} = \psi_{xx}\psi_{yy} - \psi_{xy}^2\,. \tag{63}$$

There are three cases, which correspond to the three panels in figure 7:

Elliptic: If $\det \boldsymbol{W} > 0$, then γ is imaginary and the local streamfunction has elliptic streamlines; $\boldsymbol{\xi}$ changes periodically in time and there is no exponential stretching.

Hyperbolic: If $\det \boldsymbol{W} < 0$ then γ is real and the streamfunction is locally hyperbolic. Then, as in lecture 1, material line elements will be stretched exponentially in one direction and compressed in the other.

Transitional: If $\det \boldsymbol{W} = 0$ then $|\boldsymbol{\xi}|$ grows linearly with time.

Following Okubo (1970) and Weiss (1991), the sign of $\det \boldsymbol{W}$ has been used to diagnose two-dimensional turbulence simulations (e.g., McWilliams 1984). Assuming that $\det \boldsymbol{W}$ is changing slowly in the Lagrangian frame, one argues that the result in (62) applies "quasistatically". For instance, using simulations of two-dimensional turbulence, McWilliams shows that in the core of a strong vortex $\psi_{xx}\psi_{yy} - \psi_{xy}^2 > 0$. The interpretation is that there is no exponential stretching of line elements in vortex cores, which indicates that these regions are isolated patches of laminar flow. This so-called Okubo–Weiss criterion is only a very rough guide to the stretching

[1] For a square matrix M

$$\det e^{M} = e^{\operatorname{tr} M}\,.$$

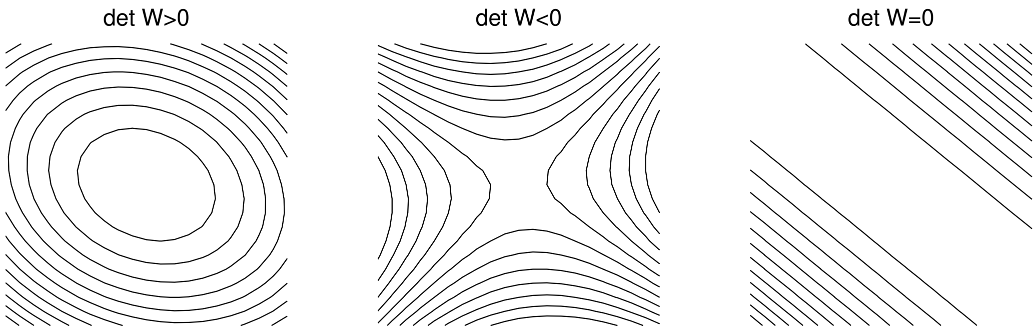

Figure 7: The sign of $\det(\boldsymbol{W}) = \psi_{xx}\psi_{yy} - \psi_{xy}^2$ determines the streamline pattern.

properties of complicated flows. The failure of the Okubo-Weiss criterion is illustrated by the random Couette process of section 1, which corresponds to the third panel of fgure 7 with $\det \boldsymbol{W} = 0$ at all time. For a further critique of the Okubo-Weiss criterion, and more refined results, see Hua and Klein (1999).

One pleasant aspect of the steady two-dimensional case is that it is possible to explicitly calculate the matrix exponential $\boldsymbol{\mathcal{J}}(t) = \exp(t\boldsymbol{W})$. (This is not the case in three dimensions.) Begin by noting that

$$\boldsymbol{W}^2 + (\det \boldsymbol{W})\boldsymbol{\mathcal{I}} = 0\,, \tag{64}$$

where $\boldsymbol{\mathcal{I}}$ is the 2×2 identity matrix. The result above is easily checked by direct evaluation, but (64) is also a consequence of $\operatorname{tr} \boldsymbol{W} = 0$ and the Cayley-Hamilton theorem. When (64) is substituted into the definition of the matrix exponential:

$$\boldsymbol{\mathcal{J}} = \exp(t\boldsymbol{W}) = \boldsymbol{\mathcal{I}} + t\boldsymbol{W} + \frac{t^2}{2}\boldsymbol{W}^2 + \frac{t^3}{6}\boldsymbol{W}^3 + \cdots \tag{65}$$

the sum collapses to

$$\boldsymbol{\mathcal{J}} = \cos\left(\sqrt{\det \boldsymbol{W}}t\right)\boldsymbol{\mathcal{I}} + \frac{\sin\left(\sqrt{\det \boldsymbol{W}}t\right)}{\sqrt{\det \boldsymbol{W}}}\boldsymbol{W}\,. \tag{66}$$

We now use the result above to formulate a renovation model.

The σ-ζ model

The "σ-ζ" model is a generalization of the random Couette process of section 1. The model is constructed using the matrix equation in (59). The idea is to define an ensemble of stretching flows in which the 2×2 matrix $\boldsymbol{W}$ is piecewise constant in the intervals $\mathcal{I}_n = \{t : (n-1)\tau < t < n\tau\}$; τ is the "decorrelation time". We use the following representation of $\boldsymbol{W}$ in the interval $\mathcal{I}_n$:

$$\boldsymbol{W}_n = \mathcal{R}_n \left[\frac{\zeta_n}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\sigma_n}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right] \mathcal{R}_n^{-1} . \tag{67}$$

where $\mathcal{R}_n$ is the rotation matrix

$$\mathcal{R}_n = \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix} . \tag{68}$$

Evaluating the matrix products gives

$$\boldsymbol{W}_n = \frac{\zeta_n}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\sigma_n}{2} \begin{pmatrix} -\cos 2\theta_n & \sin 2\theta_n \\ \sin 2\theta_n & \cos 2\theta_n \end{pmatrix} . \tag{69}$$

ζ_n is the vorticity and σ_n the strain. Isotropy is ensured by picking the random angle $0 < \theta_n < 2\pi$ from a uniform density. (We use $2\theta_n$ because the principal strain axes are at angle θ_n to the coordinate axes.)

Because $\boldsymbol{W}_n$ is constant in $\mathcal{I}_n$ the calculation of stretching rates can be reduced to a product of random matrices. The terms in the product are $\exp(\tau \boldsymbol{W}_n)$ and, using (66), one can obtain this matrix exponential analytically. There is an extensive and difficult literature devoted to calculating the statistical properties of products of random matrices (e.g., Crisanti, Paladin & Vulpiani, 1993). It is fortunate that we can avoid these complications by using the isotropy of the σ-ζ model to reduce averages of matrix products to averages of scalar products.

Two important properties of $\boldsymbol{W}_n$ are easily related to the vorticity and the strain:

$$\det \boldsymbol{W}_n = \frac{1}{4}\left(\zeta_n^2 - \sigma_n^2\right) , \qquad \mathrm{tr}\left(\boldsymbol{W}_n^{\mathrm{T}} \boldsymbol{W}_n\right) = \frac{1}{2}\left(\zeta_n^2 + \sigma_n^2\right) . \tag{70}$$

In the examples which follow we will use σ-ζ ensembles which model spatially homogeneous flows, for which $\langle \sigma^2 \rangle = \langle \zeta^2 \rangle$. In this case $\langle \det \boldsymbol{W}_n \rangle = 0$ and "on average" the Okubo-Weiss criterion is zero.

We employ (66) to obtain an explicit expression for the matrix $\boldsymbol{\mathcal{J}}_n = \exp(\tau \boldsymbol{W}_n)$. It turns out that we do not need the full details: all that is required is

$$\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\mathcal{J}}_n^{\mathrm{T}}\boldsymbol{\mathcal{J}}_n\right) = 1 + \Xi(\sigma_n, \tau_n, \tau)\,, \tag{71}$$

where

$$\Xi(\sigma, \zeta, \tau) \equiv \frac{\sigma^2}{\zeta^2 - \sigma^2}\left[1 - \cos\left(\sqrt{\zeta^2 - \sigma^2}\tau\right)\right]\,. \tag{72}$$

The "trace formula" above should be known to experts on two-dimensional stretching problems, but I have not found (71) in the literature.

The exponents γ_2 and γ_0 of the σ-ζ model

Consider the first interval $\mathcal{I}_1$, and suppose that at $t = 0$, $\boldsymbol{\xi} = \ell_0(\cos\chi, \sin\chi)$. At $t = \tau$ we have

$$\ell_1^2 = \boldsymbol{\xi}_0^{\mathrm{T}}\boldsymbol{\mathcal{J}}_1{}^{\mathrm{T}}\boldsymbol{\mathcal{J}}_1\boldsymbol{\xi}_0\,. \tag{73}$$

Now we use isotropy to average (73) over the random direction χ of the element $\boldsymbol{\xi}_0$. A trivial calculation gives

$$\langle(\ell_1/\ell_0)^2\rangle_\chi = \frac{1}{2}\mathrm{tr}\,\left(\boldsymbol{\mathcal{J}}_1{}^{\mathrm{T}}\boldsymbol{\mathcal{J}}_1\right)\,. \tag{74}$$

The RHS of (74) is given explicitly in (71). We must still average over the random variables σ and ζ. This gives

$$\langle(\ell_1/\ell_0)^2\rangle = 1 + \iint \mathcal{P}(\sigma, \zeta)\Xi(\sigma, \zeta, \tau)\,\mathrm{d}\sigma\mathrm{d}\zeta\,, \tag{75}$$

where $\mathcal{P}(\sigma, \zeta)$ is the joint PDF of σ and ζ [2]

[2] If σ and ζ are independent and identically distributed random variables then $\mathcal{P}(\sigma, \zeta) = \hat{\mathcal{P}}(\sigma)\hat{\mathcal{P}}(\zeta)$. The random Couette model of section 1 is an example with

$$\mathcal{P}(\sigma, \zeta) = \frac{1}{4}\left[\delta(\sigma + \beta) + \delta(\sigma - \beta)\right]\left[\delta(\zeta + \beta) + \delta(\zeta - \beta)\right]\,.$$

We are now well on our way to computing the rate at which ℓ^2 grows with the number of renovation cycles, n. The average stretching of ℓ^2 in each $\mathcal{I}_n$ is independent of the previous $\mathcal{I}$'s. Thus, to compute the growth of ℓ^2 over n renovation cycles, we can simply raise average ℓ^2-stretching factor in a single $\mathcal{I}$ to the n'th power:

$$\langle(\ell_n/\ell_0)^2\rangle = \left\{1 + \iint \mathcal{P}(\sigma,\zeta)\Xi(\sigma,\zeta,\tau)\,\mathrm{d}\sigma\mathrm{d}\zeta\right\}^n . \tag{76}$$

Using $n = t/\tau$, and recalling the definition of γ_p from (8), it follows that

$$\gamma_2 = \frac{1}{2\tau}\ln\left\{1 + \iint \mathcal{P}(\sigma,\zeta)\Xi(\sigma,\zeta,\tau)\,\mathrm{d}\sigma\mathrm{d}\zeta\right\} . \tag{77}$$

To further simplify the integral above we must specify the probability density function $\mathcal{P}(\sigma,\zeta)$ (examples follow).

Now we turn to γ_0. Taking the log of (73), writing $\boldsymbol{\xi}_0 = \ell_0(\cos\chi, \sin\chi)$, and then integrating[3] over χ, we have after some travail,

$$\langle\ln(\ell_1/\ell_0)\rangle_\chi = \frac{1}{2}\ln\left(1 + \frac{\Xi}{2}\right) , \tag{78}$$

where $\Xi(\sigma,\zeta,\tau)$ is given in (72). Averaging over σ and ζ, and using $\gamma_0 = \tau^{-1}\langle\ln(\ell_1/\ell_0)\rangle$, gives

$$\gamma_0 = \frac{1}{2\tau}\iint \mathcal{P}(\sigma,\zeta)\ln\left[1 + \frac{1}{2}\Xi(\sigma,\zeta,\tau)\right]\,\mathrm{d}\sigma\mathrm{d}\zeta . \tag{79}$$

The expression above should be compared with that for γ_2 in (77).

The Batchelor and Kraichnan limits

Our account of stretching exponents does not follow the historical path. The pioneering papers by Batchelor (1959) and Kraichnan (1974) considered limiting cases — slowly decorrelating in the case of Batchelor and rapidly decorrelating in the case of Kraichnan — in which stretching rates can be calculated approximately. A major advantage of these approximations is that they

[3]The integral

$$\int_0^\pi \ln(a \pm b\cos x)\,\mathrm{d}x = \pi\ln\left[\left(a + \sqrt{a^2 - b^2}\right)/2\right] ,$$

is useful.

work equally well in two and three dimensional space. On the other hand, by considering exactly soluble two-dimensional models we can extract the Batchelor and Kraichnan limits as special cases.

Batchelor (1959) considered stretching by slowly decorrelating velocity fields. This is the limit in which $\zeta\tau$ and $\sigma\tau$ are large. Batchelor's main conclusion is that in this quasisteady limit the net stretching is dominated by hyperbolic straining events. Batchelor's limit is discussed further in problem 4.1.

Kraichnan (1974) considered the opposite limit in which $\zeta\tau$ and $\sigma\tau$ are small. In this rapidly decorrelating limit we can simplify the exact expressions in (77) and (79) by noting that $\Xi \approx (\sigma\tau)^2/2 \ll 1$. This short-correlation time approximation gives

$$\gamma_0 \approx \frac{1}{8}\langle\sigma^2\rangle\tau\,, \quad \text{and} \quad \gamma_2 \approx \frac{1}{4}\langle\sigma^2\rangle\tau\,. \tag{80}$$

In this limit the exponents are independent of the vorticity and proportional to the mean square strain.

The renovating wave model again

In this section we calculate the average growth of ℓ^2 using the renovating wave (RW) model. It is interesting to see how this calculation can be done without using matrix identities such as (66).

Begin by recalling the definition of the RW model. The RW streamfunction is

$$\mathcal{I}_n = (n-1)\tau_* < t < n\tau_* : \qquad \psi_n \equiv \cos[\cos\theta_n\, x + \sin\theta_n\, y + \varphi_n]. \tag{81}$$

In (81), θ_n and φ_n are random phases and τ_* is the decorrelation time. The random phases are reinitialized at $t = n\tau_*$ so there is the complete and sudden loss of memory at these instants. (In this section we use the dimensionless version of the RW model; the parameter $\tau_* \equiv \tau kU$ is the ratio of the correlation time τ to the maximum shear of the sinusoidal wave kU.)

The renovating wave model is equivalent to the random map

$$(x_{n+1}, y_{n+1}) = (x_n, y_n) + (s_n, -c_n)\sin[c_n x_n + s_n y_n + \varphi_n]\tau\,, \tag{82}$$

where $(c_n, s_n) \equiv (\cos\theta_n, \sin\theta_n)$. The Jacobian matrix can easily be obtained by differentiation of (82):

$$\boldsymbol{\mathcal{J}}^{(n)} = e^{\tau_* \boldsymbol{W}^{(n)}} = \begin{bmatrix} 1 + c_n s_n \tau_* \psi_n & s_n^2 \tau_* \psi_n \\ -c_n^2 \tau_* \psi_n & 1 - c_n s_n \tau_* \psi_n \end{bmatrix} . \tag{83}$$

Notice that $\det \boldsymbol{\mathcal{J}}^{(n)} = 1$: the map is area preserving.

Using $\boldsymbol{\mathcal{J}}^{(n)}$ we can track the stretching of an infinitesimal material element as

$$\boldsymbol{\xi}_{n+1} = \boldsymbol{\mathcal{J}}^{(n)} \boldsymbol{\xi}_n, \qquad \Rightarrow \qquad \ell_{n+1}^2 = \boldsymbol{\xi}_{n+1}^{\mathrm{T}} \boldsymbol{\xi}_{n+1} = \boldsymbol{\xi}_n^{\mathrm{T}} \boldsymbol{\mathcal{K}}^{(n)} \boldsymbol{\xi}_n, \tag{84}$$

where $\boldsymbol{\mathcal{K}}^{(n)} = \boldsymbol{\mathcal{J}}^{(n)\mathrm{T}} \boldsymbol{\mathcal{J}}^{(n)}$. Explicitly:

$$\boldsymbol{\mathcal{K}}^{(n)} = \begin{bmatrix} (1 + c_n s_n \psi_n \tau_*)^2 + c_n^4 \psi_n^2 \tau_*^2 & (s_n^2 - c_n^2)\psi_n \tau_* + c_n s_n \psi_n^2 \tau_*^2 \\ (s_n^2 - c_n^2)\psi_n \tau_* + c_n s_n \psi_n^2 \tau_*^2 & (1 - c_n s_n \psi_n \tau_*)^2 + s_n^4 \psi_n^2 \tau_*^2 \end{bmatrix} . \tag{85}$$

To compute the stretching rate we consider an element which has length ℓ_0 at $t = 0$. Because the problem is isotropic, it is harmless to choose the coordinate system so that this element lies along the x-axis: $\boldsymbol{\xi}_0 = \ell_0(1, 0)$. After the first iteration of the map:

$$\ell_1^2 = \boldsymbol{\mathcal{K}}_{11}^{(1)} \ell_0^2 = \left[(1 + c_1 s_1 \psi_1 \tau_*)^2 + c_1^4 \psi_1^2 \tau_*^2 \right] \ell_0^2. \tag{86}$$

Averaging (86) over the phases θ_1 and φ_1 gives

$$\langle (\ell_1/\ell_0)^2 \rangle = \left(1 + \frac{\tau_*^2}{4} \right) . \tag{87}$$

If you are suspicious of the argument above, then you might prefer to align the initial material element at an arbitrary angle, say $\boldsymbol{\xi}_0 = \ell_0(\cos\chi, \sin\chi)$, and first average over χ. The result is the same.

Because each $\boldsymbol{\mathcal{J}}^{(n)}$ is independent of the earlier $\boldsymbol{\mathcal{J}}$'s the average growth of ℓ^2 is

$$\langle (\ell_n/\ell_0)^2 \rangle = \left(1 + \frac{\tau_*^2}{4} \right)^n . \tag{88}$$

Using $t = n\tau_*$, (88) can be written as

$$\langle (\ell_n/\ell_0)^2 \rangle^{1/2} = \mathrm{e}^{\gamma_2 t} , \qquad \gamma_2 \equiv \frac{1}{2\tau_*} \ln \left(1 + \frac{\tau_*^2}{4} \right) . \tag{89}$$

Aside from notional differences, γ_2 above is identical to (7).

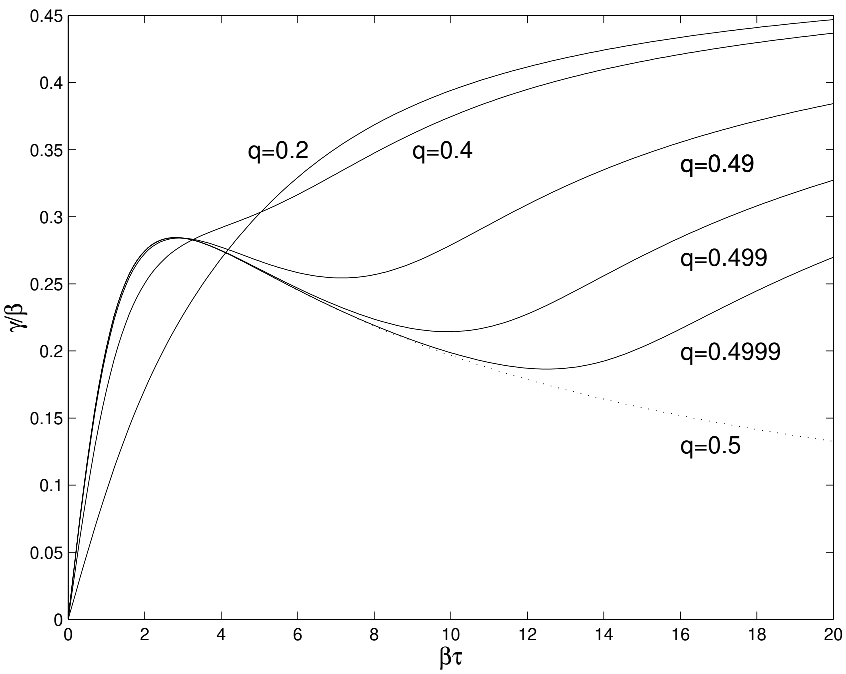

Figure 8: The nondimensional stretching exponent γ_2/β in (90) as a function of $\beta\tau$ for various values of q. If $q = 1/2$, then $\det \boldsymbol{W}$ is zero identically and $\gamma_2 \to 0$ as $\tau \to \infty$. When q is slightly less than $1/2$, and τ is sufficiently large, the occasional hyperbolic points can make a large contribution to the stretching exponent γ_2.

Problem 4.1. Suppose that σ_n and ζ_n are identical and independently distributed random variables, equal to β with probability q, $-\beta$ with probability q, or zero with probability $1 - 2q$. With this prescription there is a hyperbolic point in $\mathcal{I}_n$, as in the middle panel of figure 7, with probability $2q(1 - 2q)$. Calculate γ_2 and discuss the dependence on the parameters β, τ and q.

Solution. Enumerating and averaging over the nine possible pairs (σ_n, ζ_n) gives

$$\gamma_2 = \frac{1}{2\tau} \ln \left\{ 1 + 2q^2\beta^2\tau^2 + 2q(1 - 2q)(\cosh \beta\tau - 1) \right\} . \tag{90}$$

Figure 8 shows the nondimensional exponent γ_2/β as a function of $\beta\tau$ for various values of q. From figure 8 we conclude that while instantaneous hyperbolic points are not essential for exponential stretching, they do help, especially if the correlation time τ is long.

Problem 4.2. Using the σ-ζ model, calculate γ_p in the Kraichnan limit.

5 Amplification of concentration gradients

In this section we discuss the amplification of ∇c which occurs when a passive scalar is advected by a random flow.

Back in (27) we noted that the quantity $\boldsymbol{\xi}\cdot\nabla c$ satisfies the conservation equation

$$\frac{D}{Dt}(\boldsymbol{\xi}\cdot\nabla c) = 0. \tag{91}$$

Equation (91) enables us to use our earlier results concerning the stretching of material elements to analyze gradient amplification. In fact, using (91), we can obtain $\boldsymbol{\nabla} c$ from $\boldsymbol{\xi}$. The first step is to construct a basis by considering the following initial value problem:

$$\frac{D\boldsymbol{\xi}_k}{Dt} = (\boldsymbol{\xi}_k\cdot\nabla)\boldsymbol{u}, \qquad \text{with IC's} \quad \boldsymbol{\xi}_1(\boldsymbol{x},0) = \hat{\boldsymbol{x}}, \quad \boldsymbol{\xi}_2(\boldsymbol{x},0) = \hat{\boldsymbol{y}}, \tag{92}$$

where the unit vectors of the coordinate system are $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$. As the fluid moves, the parallelogram spanned by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ will deform. But because $\boldsymbol{u}$ is incompressible, the area of the parallelogram is constant and so

$$\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \hat{\boldsymbol{z}}, \qquad \text{(for all } t\text{)}. \tag{93}$$

If we can solve (92) for $\boldsymbol{\xi}_1$, then we can use (91) and (93) to calculate $\boldsymbol{\xi}_2$ and ∇c.

An example

As an example of this procedure, suppose that the initial condition is $c(\boldsymbol{x},0) = y$. Then it follows from (91) that:

$$\boldsymbol{\xi}_1\cdot\nabla c = 0 \qquad \text{and} \qquad \boldsymbol{\xi}_2\cdot\nabla c = 1 \qquad \text{(for all } t\text{)}. \tag{94}$$

Using (93) and (94) we see that

$$\boldsymbol{\nabla} c = \hat{\boldsymbol{z}} \times \boldsymbol{\xi}_1. \tag{95}$$

Thus, in this example, once we calculate $\boldsymbol{\xi}_1$ we obtain ∇c as a bonus.

Figure 9 displays the numerical solution for c and $|\nabla c|$ after 6 iterations of the renovating wave model. The initial condition is $c(\boldsymbol{x},0) = y$, so that

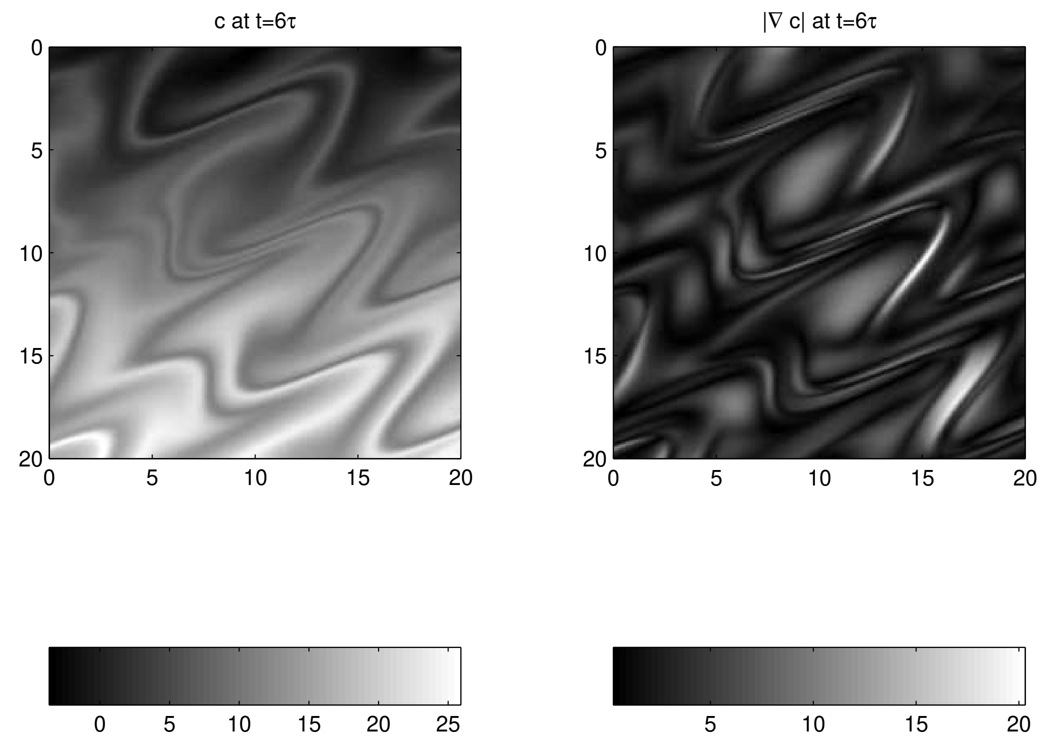

Figure 9: Numerical solution of the renovating wave model with $\tau = 2$. The initial condition is $c(x, y, 0) = y$. Already, at $t = 6\tau$, $|\nabla c|$ is greatly amplified in some regions.

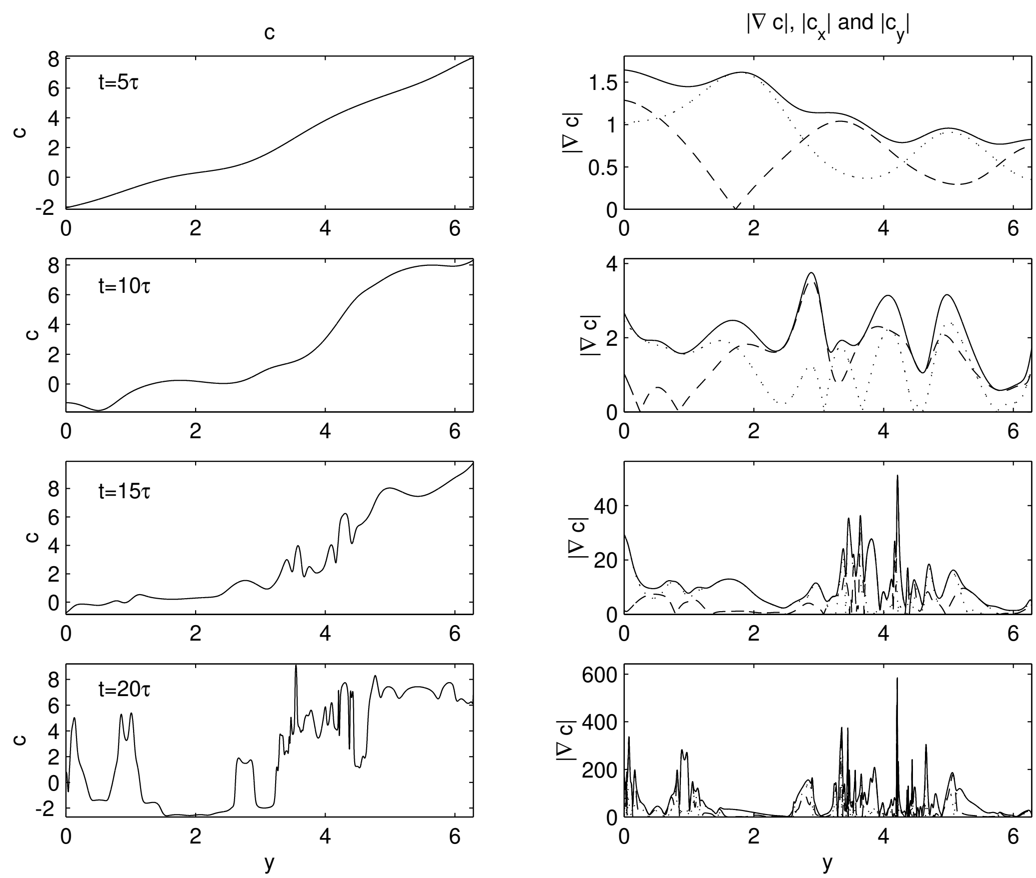

Figure 10: A numerical solution of the renovating wave model with $\tau = 1$. The initial condition is $c(x, y, 0) = y$. The plots show the values of c and $|\nabla c|$ along the slice $x = 0$. After 20 iterations, $|\nabla c|$ has developed strong spatial intermittency.

$\nabla c(\boldsymbol{x}, 0) = \hat{\boldsymbol{y}}$; the decorrelation time is $\tau = 2$. The field in figure 9 is obtained using a 256×256 grid. To find c at the grid point $\boldsymbol{x}$ at time $t = n\tau$, one iterates the renovating wave model backwards in time till the initial location (a, b) is determined, and then $c(\boldsymbol{x}, t) = b$. In parallel with this backwards iteration, $\boldsymbol{\xi}(\boldsymbol{x}, n\tau)$ is computed by matrix multiplication of the $\boldsymbol{\mathcal{J}}^{(n)}$ defined in (83), and then ∇c is given by (95).

An important feature of stirring is the development of *intermittency* in the concentration gradient, $|\nabla c|$. In figure 10 the development of intermittency is illustrated, again using the renovating wave model. After 20 iterations there are "hotspots" in which large values of $|\nabla c|$ are concentrated. Without diffusion, the gradient of c condenses onto a fractal set as the number of iterations increases (Városi, Antonsen & Ott 1991).

The filamentation transition

Discuss the interesting paper by Neufeld, López and Haynes (1999)....

6 Three dimensional incompressible flow

Can we generalize the σ-ζ model to three-dimensions, or are we limited to special cases, such as the Batchelor and Kraichnan limits? The first step is to construct at 3×3 matrix $\boldsymbol{W}$ analogous to (69). The matrix has 9 components, but because the trace is zero only eight of these are independent. Two of the eight components are equivalent to rotations in three dimensional space, and the remaining six are the principal strains $(\sigma_1, \sigma_2, \sigma_3)$ and the components of the vorticity, $(\zeta_1, \zeta_2, \zeta_3)$. In other words, we can represent an arbitrary $\boldsymbol{W}$ as

$$\boldsymbol{W} = \frac{1}{2}\mathcal{R}^{-1}\left[\begin{pmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{pmatrix} + \begin{pmatrix}0 & -\zeta_3 & \zeta_2\\ \zeta_3 & 0 & -\zeta_1\\ -\zeta_2 & \zeta_1 & 0\end{pmatrix}\right]\mathcal{R} \tag{96}$$

where $\mathcal{R}(\theta_1, \theta_2, \theta_3)$ is a random three-dimensional rotation matrix. Notice that the constraint $\sigma_1 + \sigma_2 + \sigma_3 = 0$ can be enforced by representing the σ's as

$$\sigma_1 = \nu_2 - \nu_3\,, \quad \sigma_2 = \nu_3 - \nu_1\,, \quad \sigma_3 = \nu_1 - \nu_2\,. \tag{97}$$

Some useful properties of the representation in (96) are $\nabla \times \boldsymbol{u} = [\zeta_1, \zeta_2, \zeta_3]$ and

$$\boldsymbol{u}\cdot\nabla \times \boldsymbol{u} = \frac{1}{2}\left[\sigma_1\zeta_1 x + \sigma_2\zeta_2 y + \sigma_3\zeta_3 z\right] . \tag{98}$$

Are there three-dimensional generalizations of the trace formulas? Some incomplete results. Invoking the Cayley-Hamilton theorem we know that

$$\boldsymbol{W}^3 - \frac{1}{2}\mathrm{tr}\,(\boldsymbol{W}^2)\boldsymbol{W} - \det(\boldsymbol{W})\boldsymbol{I} = 0\,, \tag{99}$$

where

$$\det(\boldsymbol{W}) = (\nu_1 - \nu_2)(\zeta_3^2 - \nu_3^2) + (\nu_2 - \nu_3)(\zeta_1^2 - \nu_1^2) + (\nu_3 - \nu_1)(\zeta_2^2 - \nu_2^2)\,, \tag{100}$$

and[4]

$$\mathrm{tr}\,(\boldsymbol{W}^2) = 2(\nu_1\nu_2 + \nu_2\nu_3 + \nu_1\nu_3 - \nu_1^2 - \nu_2^2 - \nu_3^2)\,. \tag{101}$$

We can obtain a pretty compact expression for $\boldsymbol{\mathcal{J}}(t) = \exp(t\boldsymbol{W})$ by guessing that this exponential has the form

$$\boldsymbol{\mathcal{J}} = A(t)\boldsymbol{I} + B(t)\boldsymbol{W} + C(t)\boldsymbol{W}^2\,. \tag{102}$$

The using method of undetermined coefficients on $\dot{\boldsymbol{\mathcal{J}}} = W\boldsymbol{\mathcal{J}}$ shows that

$$\dot{A} = \det(\boldsymbol{W})C\,, \quad \dot{B} = A + \frac{1}{2}\mathrm{tr}\,(\boldsymbol{W}^2)C\,, \quad \dot{C} = B\,. \tag{103}$$

All we really need $\mathrm{tr}\,(\boldsymbol{\mathcal{J}}^{\mathrm{T}}\boldsymbol{\mathcal{J}})$....

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[4]Suppose that the eignevalues of $\boldsymbol{W}$ are λ_1, λ_2 and λ_3. Then because $\sum_{i=1}^{3}\lambda_i^p = \mathrm{tr}\,(\boldsymbol{W}^p)$ we can fiddle around and show that $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = -\mathrm{tr}\,(\boldsymbol{W}^2)/2$.

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