回転球面上での 2 次元順圧流体の定式化

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この文書では回転 2 次元球面上の非圧縮流体の支配方程式と渦度方程式を導出することを行う.

導出の計算の見通しは 3 次元系において渦度方程式をベクトル形式で表現し, その後に球面への拘束条件を適用し動径成分を書き下すやり方がお薦めである (理論マニュアル線形波動「ロスビー波 (2 次元非発散球面)」) が, 以下では緯度経度動径座標の各成分の式をまず書き下し, 式変形していくという手間のかかる計算を敢えて行う.

1　回転 2 次元球面上の非圧縮流体の支配方程式の導出

まず 3 次元回転系の非圧縮流体の支配方程式から回転 2 次元球面上の非圧縮流体の支配方程式を導出する. 3 次元回転系での非圧縮流体の支配方程式を緯度経度動径座標 (λ, φ, r) で書き表すと,

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{u}{r\cos\varphi}\frac{\partial u}{\partial \lambda} + \frac{v}{r}\frac{\partial u}{\partial \varphi} + w\frac{\partial u}{\partial r} + \frac{uw - uv\tan\varphi}{r} \\
&\quad -2\Omega\sin\varphi v + 2\Omega\cos\varphi w = -\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial \lambda} \\
&\quad +\nu\left[\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \frac{\tan\varphi}{r^2}\frac{\partial u}{\partial \varphi} + \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)\right. \\
&\qquad \left. +\frac{2}{r^2\cos\varphi}\frac{\partial w}{\partial \lambda} - \frac{2\sin\varphi}{r^2\cos^2\varphi}\frac{\partial v}{\partial \lambda} - \frac{u}{r^2\cos^2\varphi},\right],
\end{aligned}
\tag{1}
$$

$$\frac{\partial v}{\partial t}+\frac{u}{r\cos\varphi}\frac{\partial v}{\partial \lambda}+\frac{v}{r}\frac{\partial v}{\partial \varphi}+w\frac{\partial v}{\partial r}+\frac{vw+u^2\tan\varphi}{r}+2\Omega\sin\varphi u=-\frac{1}{\rho r}\frac{\partial p}{\partial \varphi}$$
$$+\nu\left[\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda^2}+\frac{1}{r^2}\frac{\partial^2 v}{\partial\varphi^2}-\frac{\tan\varphi}{r^2}\frac{\partial v}{\partial\varphi}+\frac{1}{r}\frac{\partial^2}{\partial r^2}(rv)\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial w}{\partial\varphi}+\frac{2\sin\varphi}{r^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}-\frac{v}{r^2\cos^2\varphi},\right], \tag{2}$$

$$\frac{\partial w}{\partial t}+\frac{u}{r\cos\varphi}\frac{\partial w}{\partial \lambda}+\frac{v}{r}\frac{\partial w}{\partial \varphi}+w\frac{\partial w}{\partial r}-\frac{u^2+v^2}{r}-2\Omega\cos\varphi u=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\nu\left[\frac{1}{r^2\cos^2\varphi}\frac{\partial^2 w}{\partial\lambda^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\varphi^2}-\frac{\tan\varphi}{r^2}\frac{\partial w}{\partial\varphi}+\frac{1}{r}\frac{\partial^2}{\partial r^2}(rw)\right.$$
$$\left.-\frac{2}{r^2}\frac{\partial v}{\partial\varphi}+\frac{2v\tan\varphi}{r^2}-\frac{2}{r^2\cos\varphi}\frac{\partial u}{\partial\lambda}-\frac{2w}{r^2},\right], \tag{3}$$

$$\frac{1}{r\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2w)=0. \tag{4}$$

ここで u,v,w はそれぞれ速度の緯度, 経度, 動径成分である. Ω は系の回転角速度でありその回転軸は北極向きである. 球面への拘束条件として, $u,v\propto r,\ w=0$ を適用し, $r\to a$ の極限をとると, w のついた項は消え, 粘性項の $\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)=\frac{2u}{a^2}$, $\frac{1}{r}\frac{\partial^2}{\partial r^2}(rv)=\frac{2v}{a^2}$ となるので,

$$\frac{\partial u}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{v}{a}\frac{\partial u}{\partial\varphi}-\frac{uv\tan\varphi}{a}-2\Omega\sin\varphi v=-\frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial\lambda}+f_\lambda, \tag{5}$$
$$\frac{\partial v}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{v}{a}\frac{\partial v}{\partial\varphi}+\frac{u^2\tan\varphi}{a}+2\Omega\sin\varphi u=-\frac{1}{\rho a}\frac{\partial p}{\partial\varphi}+f_\varphi, \tag{6}$$
$$\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)=0. \tag{7}$$

ただし f_λ, f_φ は粘性項であり,

$$f_\lambda=\nu\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 u}{\partial\lambda^2}+\frac{1}{a^2}\frac{\partial^2 u}{\partial\varphi^2}-\frac{\tan\varphi}{a^2}\frac{\partial u}{\partial\varphi}+\frac{2u}{a^2}-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}-\frac{u}{a^2\cos^2\varphi}\right] \tag{8}$$
$$f_\varphi=\nu\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda^2}+\frac{1}{a^2}\frac{\partial^2 v}{\partial\varphi^2}-\frac{\tan\varphi}{a^2}\frac{\partial v}{\partial\varphi}+\frac{2v}{a^2}+\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}-\frac{v}{a^2\cos^2\varphi}\right] \tag{9}$$

球面への拘束条件を適用した $u,v\propto r,\ w=0$ は r が有限の元では正しい解にはならないことに注意されたい. 運動方程式の動径成分とつじつまをあわせることができなくなる. $r\to a$ の極限では動径方向には壁からの応力により球面へ運動が拘束されていると考えて運動方程式の動径成分は扱わない.

2　回転 2 次元球面上の非圧縮流体の渦度方程式の導出

ここでは回転する 2 次元球面上の非圧縮流体の支配方程式から渦度方程式を導出する. 支配方程式は回転系での 2 次元球面上の運動方程式および連続の式 (5),(6),(7) である. 渦度方程式は (5) に $-\dfrac{1}{a\cos\varphi}\dfrac{\partial}{\partial\varphi}\cos\varphi$, (6) に $\dfrac{1}{a\cos\varphi}\dfrac{\partial}{\partial\lambda}$ を作用させることにより求まる. 両式の時間変化項からは

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial u}{\partial t}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}\left[-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right]$$
$$=\frac{\partial\zeta}{\partial t}$$

ここで $\zeta=-\dfrac{1}{a\cos\varphi}\dfrac{\partial}{\partial\varphi}(u\cos\varphi)+\dfrac{1}{a\cos\varphi}\dfrac{\partial v}{\partial\lambda}$ は渦度 (の動径成分) である.

非線形項からは

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left[\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{v}{a}\frac{\partial u}{\partial\varphi}-\frac{uv\tan\varphi}{a}\right]$$
$$+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{v}{a}\frac{\partial v}{\partial\varphi}+\frac{u^2\tan\varphi}{a}\right]$$

渦度の移流の形と連続の式の形を意識して変形していく. 第 1 項目は

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}\right)=-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{u}{a}\frac{\partial u}{\partial\lambda}\right)$$
$$=-\frac{1}{a\cos\varphi}\frac{u}{a}\frac{\partial}{\partial\lambda}\left(\frac{\partial u}{\partial\varphi}\right)-\frac{1}{a^2\cos\varphi}\left(\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}\right)$$
$$=-\frac{u}{a\cos\varphi}\frac{1}{a}\frac{\partial}{\partial\lambda}\left[\frac{\partial}{\partial\varphi}\frac{u\cos\varphi}{\cos\varphi}\right]-\frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}$$
$$=-\frac{u}{a\cos\varphi}\frac{1}{a}\frac{\partial}{\partial\lambda}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{u\cos\varphi\sin\varphi}{\cos^2\varphi}\right]-\frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}$$
$$=-\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]-\frac{u}{a\cos\varphi}\frac{\tan\varphi}{a}\frac{\partial u}{\partial\lambda}-\frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}$$

第 2 項目は

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{v}{a}\frac{\partial u}{\partial\varphi}\right)=-\frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a}\frac{\partial u}{\partial\varphi}\right)-\frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)$$
$$=-\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a}\frac{\partial}{\partial\varphi}\left(\frac{u\cos\varphi}{\cos\varphi}\right)\right]-\frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)$$
$$=-\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{u\cos\varphi}{a}\frac{-\sin\varphi}{-\cos\varphi^2}\right]-\frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)$$

$$= -\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{u\tan\varphi}{a}\right] - \frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)$$

第 3 項目は

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left(-\frac{uv\tan\varphi}{a}\right) = \frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + \frac{v\cos\varphi}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{u\tan\varphi}{a}\right)$$
$$= \frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{u\tan\varphi}{a}\right)$$

第 4 項目は

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) = \frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial v}{\partial\lambda}$$

第 5 項目は

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{v}{a}\frac{\partial v}{\partial\varphi}\right) = \frac{v}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{\partial v}{\partial\varphi}$$
$$= \frac{v}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\cos\varphi}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{\partial v}{\partial\varphi}$$
$$= \frac{v}{a}\frac{1}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right) - \frac{v}{a}\frac{1}{a\cos\varphi}\left(\frac{\sin\varphi}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{\partial v}{\partial\varphi}$$
$$= \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) - \frac{v\tan\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{\partial v}{\partial\varphi}$$
$$= \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\left(\frac{\partial v}{\partial\varphi} - v\tan\varphi\right)$$
$$= \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)$$

第 6 項目は

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{u^2\tan\varphi}{a}\right) = \frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}$$

全ての項を足しあわせると,

$$\begin{aligned} = \quad & -\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{u}{a\cos\varphi}\frac{\tan\varphi}{a}\frac{\partial u}{\partial\lambda} - \frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi} \\ & -\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{u\tan\varphi}{a}\right] - \frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) \\ & +\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + \frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{u\tan\varphi}{a}\right) \\ & +\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial v}{\partial\lambda} \end{aligned}$$

$$
\begin{aligned}
&+\frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\\
&+\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}\\
=\ &-\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]-\frac{1}{a^2\cos\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial u}{\partial\varphi}\\
&-\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]-\frac{1}{a}\frac{\partial u}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\\
&+\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\\
&+\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)+\frac{1}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\frac{\partial v}{\partial\lambda}\\
&+\frac{v}{a}\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\\
&+\frac{u\tan\varphi}{a}\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}\\
=\ &\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]+\frac{v}{a}\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\\
&+\frac{u\tan\varphi}{a}\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]-\frac{1}{a}\frac{\partial u}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]\\
&+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]\\
=\ &\left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\\
&+\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a}\frac{\partial u}{\partial\varphi}+\frac{u\tan\varphi}{a}\right]\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]\\
=\ &\left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\\
&+\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]\\
=\ &\left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\zeta+\zeta\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]\\
=\ &\left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\zeta
\end{aligned}
$$

最後に連続の式 (7) を用いた.

コリオリ力の項からは

$$
-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi(-2\Omega\sin\varphi v)+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}(2\Omega\sin\varphi u)
$$

$$= 2\Omega\frac{\sin\varphi}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi) + 2\Omega\frac{v\cos\varphi}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi) + 2\Omega\frac{\sin\varphi}{a\cos\varphi}\frac{\partial u}{\partial\lambda}$$
$$= 2\Omega\sin\varphi\left[\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] + 2\Omega\frac{v}{a}\cos\varphi = 2\Omega\frac{v}{a}\cos\varphi.$$

圧力項は

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left(\frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial\lambda}\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{1}{\rho a}\frac{\partial p}{\partial\varphi}\right)$$
$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho a}\frac{\partial p}{\partial\lambda}\right) + \frac{1}{\rho a^2\cos\varphi}\frac{\partial^2 p}{\partial\lambda\partial\varphi}$$
$$= -\frac{1}{\rho a^2\cos\varphi}\frac{\partial^2 p}{\partial\lambda\partial\varphi} + \frac{1}{\rho a^2\cos\varphi}\frac{\partial^2 p}{\partial\lambda\partial\varphi} = 0.$$

粘性項は, まず f_λ の 2,3,6 項目をまとめて

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial\varphi^2} - \frac{\tan\varphi}{a^2}\frac{\partial u}{\partial\varphi} - \frac{u}{a^2\cos^2\varphi} = \frac{1}{a^2}\frac{\partial^2 u}{\partial\varphi^2} - \frac{1}{a^2}\frac{\partial}{\partial\varphi}(u\tan\varphi)$$
$$= \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left(\frac{\partial u}{\partial\varphi} - u\tan\varphi\right) = \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]$$

f_φ も同様に,

$$\frac{1}{a^2}\frac{\partial^2 v}{\partial\varphi^2} - \frac{\tan\varphi}{a^2}\frac{\partial v}{\partial\varphi} - \frac{v}{a^2\cos^2\varphi} = \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]$$

したがって,

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi f_\lambda) + \frac{1}{a\cos\varphi}\frac{\partial f_\varphi}{\partial\lambda}$$
$$= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left\{\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 u}{\partial\lambda^2} + \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] + \frac{2u}{a^2} - \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}\right\}$$
$$+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left\{\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda^2} + \frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right] + \frac{2v}{a^2} + \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right\}$$

これより渦度の形を意識しながら変形していく. 第 1 項目は

$$-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left\{\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 u}{\partial\lambda^2}\right\} = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{1}{a^2\cos\varphi}\frac{\partial^2 u}{\partial\lambda^2}\right]$$
$$= -\frac{1}{a\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\frac{\partial}{\partial\varphi}\left(\frac{u}{a^2\cos\varphi}\right) = -\frac{1}{a\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\frac{\partial}{\partial\varphi}\left(\frac{u\cos\varphi}{a^2\cos^2\varphi}\right)$$
$$= -\frac{1}{a\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi) + u\cos\varphi\frac{2\sin\varphi\cos\varphi}{a^2\cos^4\varphi}\right]$$
$$= -\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{1}{a\cos\varphi}\frac{\partial^2}{\partial\lambda^2}\left(\frac{2u\sin\varphi}{a^2\cos^2\varphi}\right)$$
$$= -\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right).$$

第 2 項目は

$$
\begin{aligned}
& -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left\{\frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\} \\
= \quad & -\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\cos\varphi\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\}.
\end{aligned}
$$

第 3 項目は

$$
-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{2u}{a^2}\right) = -\frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi).
$$

第 4 項目は

$$
-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\left(-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}\right) = \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right).
$$

第 5 項目は

$$
\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda^2}\right] = \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right].
$$

第 6 項目は

$$
\begin{aligned}
& \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left\{\frac{1}{a^2}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)\right]\right\} = \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial v}{\partial\lambda}\right)\right] \\
= \quad & \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\cos\varphi}{\cos\varphi^2}\frac{\partial v}{\partial\lambda}\right)\right] \\
= \quad & \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\left(\cos^2\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right)+\frac{1}{\cos\varphi}\frac{\partial v}{\partial\lambda}(-2\sin\varphi\cos\varphi)\right)\right] \\
= \quad & \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right)-\frac{2\sin\varphi}{\cos\varphi}\frac{\partial v}{\partial\lambda}\right] \\
= \quad & \frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)\right]-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right).
\end{aligned}
$$

第 7 項目は

$$
\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2v}{a^2}\right) = \frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}.
$$

第 8 項目はそのままで

$$
\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right).
$$

全てまとめると

$$
\begin{aligned}
& -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi f_\lambda)+\frac{1}{a\cos\varphi}\frac{\partial f_\varphi}{\partial\lambda} \\
= \; & -\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right) \\
& -\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\cos\varphi\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\} \\
& -\frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right) \\
& +\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right]+\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)\right] \\
& -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right)+\frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right) \\
= \; & -\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
& -\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left\{\cos\varphi\frac{\partial}{\partial\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right]\right\} \\
& -\frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right] \\
& +\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right)\right]+\frac{2}{a^2}\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} \\
= \; & \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
& +\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right)\right] \\
& +\frac{2}{a^2}\left[\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)\right] \\
= \; & \frac{1}{a^2\cos^2\varphi}\frac{\partial^2\zeta}{\partial\lambda^2}+\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\zeta}{\partial\varphi}\right)+\frac{2\zeta}{a^2}.
\end{aligned}
$$

これより, 渦度方程式は

$$
\begin{aligned}
& \frac{\partial\zeta}{\partial t}+\left[\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}\right]\zeta+2\Omega\frac{v}{a}\cos\varphi \\
& \qquad =\nu\left[\frac{1}{a^2\cos^2\varphi}\frac{\partial^2\zeta}{\partial\lambda^2}+\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\zeta}{\partial\varphi}\right)+\frac{2\zeta}{a^2}\right].
\end{aligned}
$$

ここで流線関数 ψ を

$$
u=-\frac{1}{a}\frac{\partial\psi}{\partial\varphi},\quad v=\frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}, \tag{10}
$$

として定義すると, 連続の式 (7) を自動的に満たす. 渦度 ζ は

$$\begin{aligned}\zeta &= -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi)+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\\ &= \frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial\psi}{\partial\varphi}\right)+\frac{1}{a^2\cos^2\varphi}\frac{\partial^2\psi}{\partial\lambda^2}=\nabla^2\psi.\end{aligned}$$

ここで $\nabla^2=\dfrac{1}{a^2\cos^2\varphi}\dfrac{\partial^2}{\partial\lambda^2}+\dfrac{1}{a^2\cos\varphi}\dfrac{\partial}{\partial\varphi}\cos\varphi\dfrac{\partial}{\partial\varphi}$ は球面上の水平ラプラシアンである. これらを渦度方程式 (10) に代入すると,

$$\frac{\partial\nabla^2\psi}{\partial t}+\left[-\frac{1}{a}\frac{\partial\psi}{\partial\varphi}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}\frac{1}{a}\frac{\partial}{\partial\varphi}\right]\nabla^2\psi+\frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda}=\nu\left(\nabla^2+\frac{2}{a^2}\right)\nabla^2\psi.$$

さらに sin 緯度 $\mu=\sin\varphi$ を導入すると, $\dfrac{\partial}{\partial\mu}=\dfrac{1}{\cos\varphi}\dfrac{\partial}{\partial\varphi}$ であるから,

$$\frac{\partial\nabla^2\psi}{\partial t}+\left[-\frac{1}{a^2}\frac{\partial\psi}{\partial\mu}\frac{\partial}{\partial\lambda}+\frac{1}{a^2}\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\mu}\right]\nabla^2\psi+\frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda}=\nu\left(\nabla^2+\frac{2}{a^2}\right)\nabla^2\psi.$$

よって

$$\frac{\partial\nabla^2\psi}{\partial t}+\frac{1}{a^2}J(\psi,\nabla^2\psi)+\frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda}=\nu\left(\nabla^2+\frac{2}{a^2}\right)\nabla^2\psi. \tag{11}$$

ただし $J(f,g)=(\partial_\lambda f)(\partial_\mu g)-(\partial_\lambda g)(\partial_\mu f)$ はヤコビアンである. ラプラシアンを μ で表しておくと

$$\nabla^2=\frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\cos\varphi\frac{\partial}{\partial\varphi}=\frac{1}{a^2(1-\mu^2)}\frac{\partial^2}{\partial\lambda^2}+\frac{1}{a^2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}. \tag{12}$$

3　角運動量保存則

ψ の球面調和函数 $Y_n^m(\mu,\lambda)$ で展開した成分のうち, $n=0$ は剛体回転流を表わす. $(n,m)=(1,0)$ は極方向を回転軸とする剛体回転流, $(n,m)=(1,1),(1,-1)$ は赤道面上の軸を回転軸とする剛体回転流である. したがってこれらの成分は各々の軸に対する角運動量となっており, 非回転系では全て, 回転系では $(n,m)=(1,0)$ 成分だけ保存しているはずである. そのことを確かめてみよう.

いま流線関数が球面調和函数の足しあわせで

$$\psi(\lambda,\mu,t)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\tilde{\psi}_{n,m}Y_n^m(\lambda,\mu)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\tilde{\psi}_{n,m}P_n^m(\lambda)e^{i\mu\lambda} \tag{13}$$

と表わされているとする. このとき ψ が実数であることから係数には

$$\tilde{\psi}_{n,-m} = \tilde{\psi}^*_{n,m} \tag{14}$$

が成り立つ.

まず $(n,m) = (1,0)$ 成分について考えよう. 渦度方程式 (11) において, 粘性項の 1,0 成分は

$$\left(\nabla^2 + \frac{2}{a^2}\right)\nabla^2\tilde{\psi}_{1,0}Y_1^0(\lambda,\mu) = \left(\frac{-1\times 2}{a^2} + \frac{2}{a^2}\right)\frac{-1\times 2}{a^2}\tilde{\psi}_{1,0}Y_1^0(\lambda,\mu) = 0. \tag{15}$$

(11) の β 項の 1,0 成分は

$$\frac{2\Omega}{a^2}\frac{\partial}{\partial\lambda}\tilde{\psi}_{1,0}Y_1^0(\lambda,\mu) = 0. \tag{16}$$

非線形項からの寄与は, (13) をヤコビアンに代入して $Y_1^0(\lambda,\mu) = \mu$ をかけて球面上で積分することにより得られる.

$$\begin{aligned}
&\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \frac{1}{a^2} J(\psi,\nabla^2\psi)Y_1^0(\lambda,\mu) \\
=\ & \frac{1}{a^2}\sum_{n,m}\sum_{n',m'}\int_0^{2\pi}\int_{-1}^{1}\left[im\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\mu Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right]d\lambda d\mu \\
=\ & \frac{1}{a^2}\sum_{n,m}\sum_{n',m'}\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\int_0^{2\pi}\int_{-1}^{1}\left[im\mu Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right]d\lambda d\mu
\end{aligned}$$

ここで $m = -m'$ 成分のみ 0 でないことに注意すると

$$\begin{aligned}
&\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu\left[im\mu Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] \\
=\ & \int_0^{2\pi}\left[im\mu Y_n^m Y_{n'}^{m'}\right]_{-1}^{1}d\lambda - \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_{n'}^{m'}d\lambda d\mu - \int_0^{2\pi}\int_{-1}^{1} im\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}d\lambda d\mu \\
&+\int_0^{2\pi}\int_{-1}^{1}\left[-im'\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right]d\lambda d\mu \\
=\ & \int_0^{2\pi}\left[im\mu Y_n^m Y_{n'}^{-m}\right]_{-1}^{1}d\lambda - \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_{n'}^{-m}d\lambda d\mu - \int_0^{2\pi}\int_{-1}^{1} im\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{-m}d\lambda d\mu \\
&+\int_0^{2\pi}\int_{-1}^{1} im\mu\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{-m}d\lambda d\mu \\
=\ & \int_0^{2\pi}\left[im\mu Y_n^m Y_{n'}^{-m}\right]_{-1}^{1}d\lambda - \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_{n'}^{-m}d\lambda d\mu \\
=\ & \int_0^{2\pi}\left[im\mu Y_n^m Y_{n'}^{-m}\right]_{-1}^{1}d\lambda - \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_n^{-m}d\lambda d\mu
\end{aligned}$$

最後に Y_n^m の直交関係を用いている. 第 1 項目の値を評価する. Y_n^m の $\mu = \pm 1$ での値は $m \neq 0$ のとき 0 である. $m = 0$ のときは係数 m がかかっているので結局この項は 0 となる. したがって, n, m, n', m' 成分からは

$$-\tilde{\psi}_{n,m}\tilde{\psi}_{n,-m}\int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_n^{-m} d\lambda d\mu = -|\tilde{\psi}_{n,m}|^2 \int_0^{2\pi}\int_{-1}^{1} imY_n^m Y_n^{-m} d\lambda d\mu \quad (17)$$

一方, $n, -m$ から

$$-|\tilde{\psi}_{n,-m}|^2 \int_0^{2\pi}\int_{-1}^{1} (-imY_n^{-m} Y_n^m) d\lambda d\mu = |\tilde{\psi}^*_{n,m}|^2 \int_0^{2\pi}\int_{-1}^{1} imY_n^{-m} Y_n^m d\lambda d\mu \quad (18)$$

したがってこの項は互いにキャンセルする. 結局非線形項からの $(n, m) = (1, 0)$ への寄与は 0 となる. 最後に時間変化項は

$$\frac{\partial}{\partial t}\nabla^2\tilde{\psi}_{1,0}(t)Y_1^0 = -\frac{1(1+1)}{a^2}\frac{d\tilde{\psi}_{1,0}}{dt}Y_1^0 \quad (19)$$

よって

$$\frac{d\tilde{\psi}_{1,0}}{dt} = 0 \quad (20)$$

となり極方向を軸とする角運動量は保存する.

次に $(n, m) = (1, 1)$ 成分について考えよう. 粘性項の $(1, 1)$ 成分は $(n, m) = (1, 0)$ と同様にして 0 である. (11) の β 項の 1,1 成分は

$$\frac{2\Omega}{a^2}\frac{\partial}{\partial\lambda}\tilde{\psi}_{1,1}Y_1^1(\lambda,\mu) == i\frac{2\Omega}{a^2}\tilde{\psi}_{1,1}Y_1^1(\lambda,\mu). \quad (21)$$

非線形項からの寄与は, (13) をヤコビアンに代入して $Y_1^1(\lambda,\mu) = \sqrt{1-\mu^2}e^{i\lambda}$ をかけて球面上で積分することにより得られる.

$$\begin{aligned}
&\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \frac{1}{a^2} J(\psi, \nabla^2\psi) Y_1^1(\lambda,\mu) \\
= \;& \frac{1}{a^2}\sum_{n,m}\sum_{n',m'}\int_0^{2\pi}\int_{-1}^{1}\left[im\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu}\right. \\
&\qquad\left. -im'\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] d\lambda d\mu \\
= \;& \frac{1}{a^2}\sum_{n,m}\sum_{n',m'}\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\int_0^{2\pi}\int_{-1}^{1}\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] d\lambda d\mu
\end{aligned}$$

ここで $m' = -1 - m$ 成分のみ 0 でないことに注意すると, $m = n$ のときは 0 になり, $0 < m < n + 1$ なる n, m に対して

$$\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right]$$

$$
\begin{aligned}
&= \int_0^{2\pi}\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m Y_{n'}^{m'}\right]_{-1}^{1}d\lambda - \int_0^{2\pi}\int_{-1}^{1} -im\frac{\mu}{\sqrt{1-\mu^2}}e^{i\lambda}Y_n^m Y_{n'}^{m'}d\lambda d\mu \\
&\quad -\int_0^{2\pi}\int_{-1}^{1} im\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}d\lambda d\mu + \int_0^{2\pi}\int_{-1}^{1}\left[-im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right]d\lambda d\mu \\
&= \int_0^{2\pi}\int_{-1}^{1} im\frac{\mu}{\sqrt{1-\mu^2}}e^{i\lambda}Y_n^m Y_{n'}^{m'}d\lambda d\mu - i(m+m')\int_0^{2\pi}\int_{-1}^{1}\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}d\lambda d\mu \\
&= \int_0^{2\pi}\int_{-1}^{1} im\frac{\mu}{\sqrt{1-\mu^2}}e^{i\lambda}Y_n^m Y_{n'}^{m'}d\lambda d\mu + i\int_0^{2\pi}\int_{-1}^{1}\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}d\lambda d\mu \\
&= i\int_0^{2\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-\mu^2}}e^{i\lambda}\left[m\mu Y_n^m + (1-\mu^2)\frac{\partial Y_n^m}{\partial\mu}\right]Y_{n'}^{m'}d\lambda d\mu
\end{aligned}
$$

ここでルジャンドル函数の漸化式

$$(1-\mu^2)\frac{\partial P_n^m}{\partial\mu} = \sqrt{1-\mu^2}P_n^{m+1} - m\mu P_n^m \tag{22}$$

より

$$
\begin{aligned}
e^{i\lambda}\left[m\mu Y_n^m + (1-\mu^2)\frac{\partial Y_n^m}{\partial\mu}\right] &= e^{i(m+1)\lambda}\left[m\mu P_n^m + (1-\mu^2)\frac{\partial P_n^m}{\partial\mu}\right] \\
&= \sqrt{1-\mu^2}P_n^{m+1}e^{i(m+1)\lambda} = \sqrt{1-\mu^2}Y_n^{m+1}.
\end{aligned}
$$

したがって, n, m 成分からの寄与は

$$
\begin{aligned}
&\frac{1}{a^2}\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\int_0^{2\pi}\int_{-1}^{1}d\lambda d\mu\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] \\
&= i\frac{1}{a^2}\tilde{\psi}_{n,m}\tilde{\psi}_{n',-m-1}\int_0^{2\pi}\int_{-1}^{1}Y_n^{m+1}Y_{n'}^{m'}d\lambda d\mu d\lambda d\mu \\
&= i\frac{1}{a^2}\tilde{\psi}_{n,m}\tilde{\psi}_{n,-m-1}\int_0^{2\pi}\int_{-1}^{1}Y_n^{m+1}Y_n^{-m-1}d\lambda d\mu d\lambda d\mu
\end{aligned}
$$

最後に直交関係を使って $n' = n$ とした.

一方 $m < 0$ の成分に対して, $m \to -m-1 > -n-1$ を選んで計算すると $m' = m$ の時のみ 0 でなく,

$$
\begin{aligned}
&\int_0^{2\pi}\int_{-1}^{1}d\lambda d\mu\left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'}\right] \\
&= \int_0^{2\pi}\int_{-1}^{1}d\lambda d\mu\left[i(-m-1)\sqrt{1-\mu^2}e^{i\lambda}Y_n^{-m-1}\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^{-m-1}}{\partial\mu}Y_{n'}^{m'}\right] \\
&= \int_0^{2\pi}\int_{-1}^{1}d\lambda d\mu\left[i(-m-1)\sqrt{1-\mu^2}e^{i\lambda}Y_n^{-m-1}\frac{\partial Y_{n'}^{m}}{\partial\mu} - im\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^{-m-1}}{\partial\mu}Y_{n'}^{m}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_{n'}^{m}\frac{\partial Y_n^{-m-1}}{\partial \mu} - i(-m-1)\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_{n'}^{m}}{\partial \mu}Y_n^{-m-1}\right] \\
&= -i\int_0^{2\pi}\int_{-1}^{1} Y_{n'}^{m+1}Y_n^{-m-1} d\lambda d\mu d\lambda d\mu \\
&= -i\int_0^{2\pi}\int_{-1}^{1} Y_{n}^{m+1}Y_n^{-m-1} d\lambda d\mu d\lambda d\mu.
\end{aligned}
$$

最後の計算は n, m の時の式変形と同様に行った. したがって, $n, -m-1$ 成分からの寄与は

$$
\begin{aligned}
&\frac{1}{a^2}\tilde{\psi}_{n,-m-1}\tilde{\psi}_{n',m'}\int_0^{2\pi}\int_{-1}^{1} d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_{n}^{m}\frac{\partial Y_{n'}^{m'}}{\partial \mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_{n}^{m}}{\partial \mu}Y_{n'}^{m'}\right] \\
&= -i\frac{1}{a^2}\tilde{\psi}_{n,-m-1}\tilde{\psi}_{n',m}\int_0^{2\pi}\int_{-1}^{1} Y_{n}^{-m-1}Y_{n'}^{m} d\lambda d\mu d\lambda d\mu \\
&= -i\frac{1}{a^2}\tilde{\psi}_{n,-m-1}\tilde{\psi}_{n,m}\int_0^{2\pi}\int_{-1}^{1} Y_{n}^{-m-1}Y_{n}^{m} d\lambda d\mu d\lambda d\mu
\end{aligned}
$$

よって n, m 成分と $n, -m-1$ 成分からの寄与が打ち消しあって非線形項は 0 となる.

最後に時間変化項は $(1,0)$ 成分と同じであり

$$
\frac{\partial}{\partial t}\nabla^2\tilde{\psi}_{1,1}(t)Y_1^1 = -\frac{1(1+1)}{a^2}\frac{d\tilde{\psi}_{1,1}}{dt}Y_1^1 \tag{23}
$$

よって $(1,1)$ 成分の式は

$$
-\frac{2}{a^2}\frac{d\tilde{\psi}_{1,1}}{dt} + i\frac{2\Omega}{a^2}\tilde{\psi}_{1,1} = 0, \qquad \tilde{\psi}_1^1 = Ae^{i\Omega t}. \tag{24}
$$

A は任意定数であり初期値で定まる. したがって $(1,1)$ 成分の振舞は

$$
\psi(\mu,\lambda,t) = Ae^{i(\lambda+\Omega t)}P_1^1(\mu) \tag{25}
$$

緯度方向に速度 Ω で逆向きに伝播する. この解は回転系で見れば波数 $(1,1)$ のロスビー波である. 慣性系から見て保存している極をとおる剛体回転流を, 回転系から見ると Ω で逆まわりしているように見えるのがこの解である.

$(1,-1)$ 成分は $(1,1)$ 成分と同様に粘性項, 非線形項の寄与が 0 となる.

$$
-\frac{2}{a^2}\frac{d\tilde{\psi}_{1,-1}}{dt} - i\frac{2\Omega}{a^2}\tilde{\psi}_{1,-1} = 0, \qquad \tilde{\psi}_1^1 = Ae^{-i\Omega t}. \tag{26}
$$

したがって $(1,1)$ 成分の振舞は

$$
\psi(\mu,\lambda,t) = Ae^{-i(\lambda+\Omega t)}P_1^1(\mu) \tag{27}
$$

緯度方向に速度 Ω で逆向きに伝播する. 特に回転がなければ時間微分が 0 となり保存する.

文献

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